Theorems Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

Multiple Choice Solutions

1. B (1998 AB4)
   B is false because this special case of the MVT called Rolle's Theorem also requires that $f(a) = f(b)$. A is true by MVT; C and D are true by EVT, E is true by the definition of a definite integral.

2. C (1988 AB20)
   Rolle's Theorem guarantees at least one value of x between a and b such that $f'(x) = 0$.

3. B (1985 BC13)
   Show $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ using MVT: $3x^2 - 6x = \dfrac{27 - 27}{3 - 0}$; $3x(x-2) = 0$ when $x = 0$ and $x = 2$; however, only 2 is eligible since there is an endpoint at $x = 0$.

4. A (1998 AB26)
   Any value of k less than $\frac{1}{2}$ will require the function to assume the value of $\frac{1}{2}$ at least twice because of the Intermediate Value Theorem on the intervals $[0, 1]$ and $[1, 2]$, so $k = 0$ is the only option.

5. D (1973 BC18 appropriate for AB)
   D could be false, consider $g(x) = 1 - x$ on $[0, 1]$.
   A is true by the Extreme Value Theorem.
   B is true because g is a function.
   C is true by the Intermediate Value Theorem.
   E is true because g is continuous.

6. D (1993 AB18)

$$f'(c) = \frac{1}{2}\cos\left(\frac{c}{2}\right);\ f'(c) = \frac{f\left(\frac{3\pi}{2}\right) - f\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2} - \frac{\pi}{2}} = \frac{\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2} - \frac{\pi}{2}} = \frac{0}{\pi} = 0;$$

$$\frac{1}{2}\cos\left(\frac{c}{2}\right) = 0;\ c = \pi$$

7. E (1993 BC44 appropriate for AB)
By the Intermediate Value Theorem, there is a c satisfying $a < c < b$ such that $f(c)$ is equal to the average value of f on the interval $[a, b]$. The average value is also given by $\frac{1}{b-a}\int_a^b f(x)\,dx$. Equating the two gives option E.

Alternatively, let $F(t) = \int_a^t f(x)\,dx$. By the Mean Value Theorem, there is a c satisfying $a < c < b$ such that $\frac{F(b)-F(a)}{b-a} = F'(c)$. $F(b)-F(a) = \int_a^b f(x)\,dx$ and $F'(c) = f(c)$ by the Fundamental Theorem of Calculus. This again gives option E as the answer. This result is called the Mean Value Theorem for Integrals.

8. B (1969 BC3 appropriate for AB)
$y = \sqrt{x}$, so $y' = \frac{1}{2\sqrt{x}}$. By the Mean Value Theorem, $\frac{1}{2\sqrt{x}} = \frac{2-0}{4-0}$, so $c = 1$. The point is $(1, 1)$.

9. E (1998 AB91)
I and III are true by IVT; II is true by MVT.

10. E (2008 AB89/BC89)
Since there is no c for which $f'(c) = 0$, Rolle's Theorem is violated and $f'(k)$ does not exist on $(-2, 2)$.

11. B (2003 AB80)
B could be false since this is a special case of MVT (Rolle's Theorem) which also requires that $f(a) = f(b)$. A and C are true by IVT; D is true by MVT; E is true by EVT.

12. D (1997 BC81 appropriate for AB)
f assumes every value between -1 and 3 on the interval $(-3, 6)$, so $f(c) = 1$ at least once.

13. 1999 AB3/BC3b

(b) Yes; Since $R(0) = R(24) = 9.6$, the Mean Value Theorem guarantees that there is a t, $0 < t < 24$, such that $R'(t) = 0$.	2 { 1: answer 1: MVT or equivalent

14. 2009B AB3bc

(b) $\dfrac{f(6) - f(a)}{6 - a} = 0$ when $f(a) = f(6)$. There are two values of a for which this is true.	2 { 1: expression for average rate of change 1: answer with reason
(c) Yes, $a = 3$. The function f is differentiable on the interval $3 < x < 6$ and continous on $3 \le x \le 6$. Also, $\dfrac{f(6) - f(3)}{6 - 3} = \dfrac{1 - 0}{6 - 3} = \dfrac{1}{3}$. By the Mean Value Theorem, there is a value c, $3 < x < 6$, such that $f'(c) = \dfrac{1}{3}$.	2 { 1: answers "yes" and identifies $a = 3$ 1: justification

15. 2008B AB5/BC5d

(d) $\dfrac{g'(7) - g'(-3)}{7 - (-3)} = \dfrac{1 - (-4)}{10} = \dfrac{1}{2}$ No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.	2 { 1: average value of $g'(x)$ 1: answer "No" with reason

16. 2007 AB 3ab

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$ $h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$ Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.	2: 1: $h(1)$ and $h(3)$ 1: conclusion, using IVT
(b) $\frac{h(3)-h(1)}{3-1} = \frac{-7-3}{3-1} = -5$ Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.	2: 1: $\frac{h(3)-h(1)}{3-1}$ 1: conclusion, using MVT

17. 2007B AB 6abd

(a) The Mean Value Theorem guarantees that there is a value c, with $2 < c < 5$, so that $f'(c) = \frac{f(5)-f(2)}{5-2} = \frac{2-5}{5-2} = -1$.	2: 1: $\frac{f(5)-f(2)}{5-2}$ 1: conclusion, using MVT
(b) $g'(x) = f'(f(x)) \cdot f'(x)$ $g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2)$ $g'(5) = f'(f(5)) \cdot f'(5) = f'(2) \cdot f'(5)$ Thus, $g'(2) = g'(5)$. Since f is twice-differentiable, g' is differentiable everywhere, so the Mean Value Theorem applied to g' on [2, 5] guarantees there is a value k, with $2 < k < 5$, such that $g''(k) = \frac{g'(5)-g'(2)}{5-2} = 0$.	3: 1: $g'(x)$ 1: $g'(2) = f'(5) \cdot f'(2) = g'(5)$ 1: uses MVT with g'
(d) Let $h(x) = f(x) - x$. $h(2) = f(2) - 2 = 5 - 2 = 3$ $h(5) = f(5) - 5 = 2 - 5 = -3$ Since $h(2) > 0 > h(5)$, the Intermediate Value Theorem guarantees that there is a value r, with $2 < r < 5$, such that $h(r) = 0$	2: 1: $h(2)$ and $h(5)$ 1: conclusion, using IVT

18. 2002 AB 6c

(c) By the Mean Value Theorem there is a c with $0 < c < 0.5$ such that $f''(c) = \frac{f'(0.5) - f'(0)}{0.5 - 0} = \frac{3 - 0}{0.5} = 6 = r$	2 { 1: reference to MVT for f' (or differentiability of f') 1: value of r for interval $0 \le x \le 0.5$