Tangent Lines and Linear Approximations Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

Multiple Choice Answers:

1. C (2003 AB16/BC16)
   Using the point-slope form of the equation of the tangent line and the point of tangency:
   $y-7=(f'(1))(x-1)$
   $-2-7=(f'(1))(-2-1)$
   $-9=-3f'(1);\ f'(1)=3.$

2. C (1997 AB10)
   $y'=-2\sin(2x)=-2$
   When $x=\frac{\pi}{4}$, $y=\cos\left(2\left(\frac{\pi}{4}\right)\right)=0$; therefore, the equation of the tangent line is
   $y-0=-2\left(x-\frac{\pi}{4}\right)$.

3. E (2003 AB89)
   $g(2)=2\cdot f(2)=2\cdot 3=6$
   $g'(x)=xf'(x)+f(x)$ (product rule);
   $g'(2)=2f'(2)+f(2)=2(-5)+3=-7$.
   $y-6=-7(x-2)$
   The only answer choice with slope of -7 is answer E.

4. A (2008 AB11)
   Using the product rule and factoring:
   $f'(x)=-6x(1-2x)^2+(1-2x)^3=(1-2x)^2(-6x+1-2x)$
   When $x=1$, $f'(1)=-7$ giving $y+1=-7(x-1)$
   The slope-intercept form of the equation of the tangent line is answer A.

5. B (1993 AB7)
   Using the quotient rule, $y'=\dfrac{(3x-2)(2)-(2x+3)(3)}{(3x-2)^2}$.
   When $x=1$, $y'(1)=-13$ so the point-slope form of the tangent line equation is
   $y-5=-13(x-1)$ which is equivalent to the standard form in answer B.

6. C (1997 AB14)
The tangent line is $y-2=5(x-3)$.
Approximate the zero when $x=3$.
$-2=5(x-3)$
$x=\frac{13}{5}=2.6$

7. B (1997 AB12)
The slope of the line, $y=\frac{1}{2}x-\frac{3}{4}$, is $m=\frac{1}{2}$.
Given $y=\frac{1}{2}x^2$, the derivative is $y'=x$.
Set $x=\frac{1}{2}$ (slope of the parallel line $y=\frac{1}{2}x-\frac{3}{4}$).
Substituting $\frac{1}{2}$ for x in the original equation, $y=\frac{1}{2}\left(\frac{1}{2}\right)^2=\frac{1}{8}$.
Therefore, answer B gives the correct point $\left(\frac{1}{2},\frac{1}{8}\right)$.

8. B (1969 AB36/BC36)
When $x=0$, $y(0)=\sqrt{4+\sin(0)}=2$
$y'=\frac{1}{2}(4+\sin x)^{-\frac{1}{2}}(\cos x)$
$y'(0)=\frac{1}{4}$
The equation of the tangent line is $y-2=\frac{1}{4}(x-0)$.
Use this tangent line to approximate: $y\approx\frac{1}{4}(0.12)+2=2.03$.

9. B (2003 AB26)
Use implicit differentiation: $6yy'-4x=-(2xy'+2y)$.
Evaluate at $(3,2)$: $6(2)y'-4(3)=-(2(3)y'+2(2))$

$$18y'=8$$
$$y'=\frac{4}{9}$$

10. A (1993 BC17)
Using implicit differentiation:
$\ln(xy) = x$
$\ln x + \ln y = x$
$\frac{1}{x} + \frac{1}{y} y' = 1$
Evaluate the original equation when $x = 1$, $\ln(y) = 1$ yields $y = e$.
Therefore, $1 + \frac{1}{e} y' = 1$ which yields the slope of the tangent line, $y' = 0$.

11. A (1988 BC11 appropriate for AB)
The slope of the normal line: $y = -\frac{1}{7}x + \frac{29}{7}$ is $m = -\frac{1}{7}$.
Since normal lines are perpendicular to tangent lines at the point of tangency, the slope of the tangent line is its negative reciprocal, so $f'(1) = 7$.

12. A (1997 AB80)
$f'(x) = 8x(2e^{4x^2})$
The slope of the tangent line is equal to 3 when $f'(x) = 3$. Use the graphing calculator to determine the *x* value when $8x(2e^{4x^2}) = 3$. Student may also use the derivative functions to graph f' without calculating the derivative by hand.

13. D (1998 AB87)
Using the graphing calculator, determine the *x* value when $f'(x) = 1$ and store the value in $A \approx 0.2367327012$. Evaluate $f(A) \approx 0.1152254911$ and store in *B*.
The equation of the tangent line will be $y - B = 1(x - A)$ or $y = x - 0.122$
*storing the values in this question is not necessary to obtain the correct answer, but is a good practice for students.

14. C (1998 AB77)
The functions f and g have parallel tangent lines when $f'(x) = g'(x)$.
Students can take the derivatives by hand, $6e^{2x} = 18x^2$, or use the calculator to solve.
$x = 0.391$.

Free Response

15. 2010 AB6ab

(a) $f'(1) = \left.\dfrac{dy}{dx}\right\|_{(1,\,2)} = 8$ An equation of the tangent line is $y = 2 + 8(x-1)$.	$2\begin{cases} 1: & f'(1) \\ 1: & \text{answer} \end{cases}$
(b) $f(1.1) \approx 2.8$ Since $y = f(x) > 0$ on the interval $1 \le x \le 1.1$, $\dfrac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right) > 0$ on this interval. Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.	$2\begin{cases} 1: & \text{approximation} \\ 1: & \text{conclusion with explanation} \end{cases}$

16. 2002 AB6b

(b) $y = 5(x-1) - 4$ $f(1.2) \approx 5(0.2) - 4 = -3$ The approximation is less than $f(1.2)$ because the graph of f is concave up on the interval $1 < x < 1.2$.	$3 \begin{cases} 1: \text{ tangent line} \\ 1: \text{ computes } y \text{ on the tangent line at } x = 1.2 \\ 1: \text{ answer with reason} \end{cases}$

17. 1996 AB6

(a) let Q be $\left(a,\ a-\dfrac{a^2}{500}\right)$

$$\frac{dy}{dx}=1-\frac{x}{250}$$

setting slopes equal:

$$1-\frac{a}{250}=\frac{\left(a-\dfrac{a^2}{500}\right)-20}{a}$$

$$a=100$$

or

$$\frac{dy}{dx}=1-\frac{x}{250}$$

equation for l : $y=\left(1-\dfrac{a}{250}\right)x+20$

setting y-values equal:

$$\left(1-\frac{a}{250}\right)a+20=a-\frac{a^2}{500}$$

$$a=100$$

4 points:
- 1: slope of tangent line from parabola
- 1: uses the condition that $\left(a,\ a-\dfrac{a^2}{500}\right)$ is on line l
- 1: uses the condition that slopes are equal at Q
- 1: answer
- 0/1 if student is solving an irrelevant equation

(b) $y=\dfrac{3}{5}x+20$

2 points:
- 1: slope
- 0/1 if $m\le 0$
- 1: equation

(c) height of hill at $x=250$:

$$y=250-\frac{250^2}{500}=125 \text{ feet}$$

height of line at $x=250$:

$$y=\frac{3}{5}(250)+20=170 \text{ feet}$$

Yes, the spotlight hits the tree since the height of the line is less than the height of the hill + tree which is 175 feet.

3 points:
- 1: height of hill
- 1: height of line
- 0/1 if height <0
- 1: answer with analysis

18. 2005B AB5/BC 5

(a) $2yy' = y + xy'$ $(2y - x)y' = y$ $y' = \dfrac{y}{2y - x}$	2	$\begin{cases} 1: & \text{implicit differentiation} \\ 1: & \text{solves for } y' \end{cases}$
(b) $\dfrac{y}{2y - x} = \dfrac{1}{2}$ $2y = 2y - x$ $x = 0$ $y = \pm\sqrt{2}$ $\left(0, \sqrt{2}\right), \left(0, -\sqrt{2}\right)$	2	$\begin{cases} 1: & \dfrac{y}{2y - x} = \dfrac{1}{2} \\ 1: & \text{answer} \end{cases}$
(c) $\dfrac{y}{2y - x} = 0$ $y = 0$ The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any x.	2	$\begin{cases} 1: & y = 0 \\ 1: & \text{explanation} \end{cases}$