Fundamental Theorem of Calculus

Students should be able to:

- Use the fundamental theorem to evaluate definite integrals

$$\int_a^b f(x)dx = F(b) - F(a)$$

- Use various forms of the fundamental theorem in application situations.

$$\int_a^b f'(x)dx = f(b) - f(a)$$

$$f(a) + \int_a^b f'(x)dx = f(b)$$

- Calculate the average value of a function over a particular interval.

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a} = \frac{F(b) - F(a)}{b-a}$$

- Use the other fundamental theorem

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))\,g'(x)$$

Multiple Choice

1. (calculator not permitted)

   $\int_1^2 \frac{x^2-1}{x+1}\,dx =$

   (A) $\frac{1}{2}$
   (B) 1
   (C) 2
   (D) $\frac{5}{2}$
   (E) ln 3

2. (calculator not permitted)

   If $\int_0^k \left(2kx - x^2\right)dx = 18$, then $k =$

   (A) −9
   (B) −3
   (C) 3
   (D) 9
   (E) 18

3. (calculator not permitted)

   What is the average value of y for the part of the curve $y = 3x - x^2$ which is in the first quadrant?

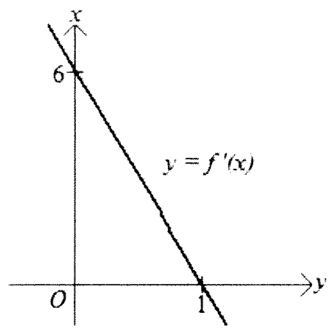
   (A) −6
   (B) −2
   (C) $\frac{3}{2}$
   (D) $\frac{9}{4}$
   (E) $\frac{9}{2}$

4. (calculator not permitted)

   If the function f has a continuous derivative on $[0, c]$, then $\int_0^c f'(x)\,dx$

   (A) $f(c)-f(0)$
   (B) $|f(c)-f(0)|$
   (C) $f(c)$
   (D) $f(x)+c$
   (E) $f''(c)-f''(0)$

5. (calculator not permitted)



   The graph of f', the derivative of f, is the line shown in the figure above. If $f(0) = 5$, then $f(1) =$

   (A) 0
   (B) 3
   (C) 6
   (D) 8
   (E) 11

6. (calculator not permitted)

   $\frac{d}{dx}\int_0^{x^2} \sin(t^3)\,dt =$

   (A) $-\cos(x^6)$
   (B) $\sin(x^3)$
   (C) $\sin(x^6)$
   (D) $2x\sin(x^3)$
   (E) $2x\sin(x^6)$

7. (calculator not permitted)

Let $f(x)=\int_{-2}^{x^2-3x} e^{t^2}\,dt$. At what value of x is $f(x)$ a minimum?

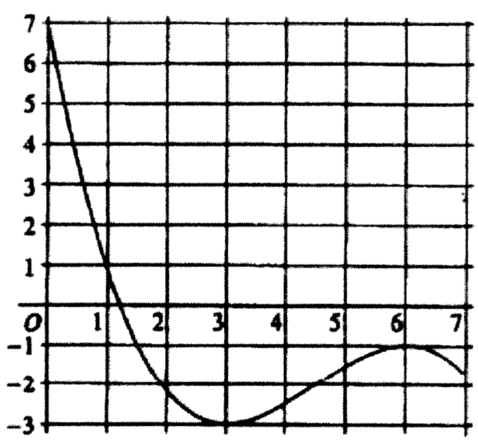
(A) For no value of x
(B) $\frac{1}{2}$
(C) $\frac{3}{2}$
(D) 2
(E) 3

8. (calculator not permitted)

$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1+\sin\theta}}\,d\theta =$

(A) $-2\left(\sqrt{2}-1\right)$
(B) $-2\sqrt{2}$
(C) $2\sqrt{2}$
(D) $2\left(\sqrt{2}-1\right)$
(E) $2\left(\sqrt{2}+1\right)$

9. (calculator not permitted)



Graph of f

The graph of the function f shown in the figure above has horizontal tangents at $x=3$ and $x=6$. If $g(x)=\int_0^{2x} f(t)\,dt$, what is the value of $g'(3)$?

(A) 0
(B) -1
(C) -2
(D) -3
(E) -6

10. (calculator not allowed)
    If p is a polynomial of degree n, $n > 0$, what is the degree of the polynomial
    $Q(x) = \int_0^x p(t)\,dt$?

    (A) 0
    (B) 1
    (C) $n - 1$
    (D) n
    (E) $n + 1$

11. (calculator not allowed)
    Suppose $g'(x) < 0$ for all $x \geq 0$ and $F(x) = \int_0^x t\, g'(t)\, dt$ for all $x \geq 0$. Which of the following statements is FALSE?

    (A) F takes on negative values.
    (B) F is continuous for all $x > 0$.
    (C) $F(x) = x\,g(x) - \int_0^x g(t)\,dt$
    (D) $F'(x)$ exists for all $x > 0$.
    (E) F is an increasing function.

12. (calculator not allowed)
    $\frac{d}{dx}\left(\int_0^{x^3} \ln\left(t^2 + 1\right)dt\right) =$

    (A) $\frac{2x^3}{x^6 + 1}$
    (B) $\frac{3x^2}{x^6 + 1}$
    (C) $\ln\left(x^6 + 1\right)$
    (D) $2x^3 \ln\left(x^6 + 1\right)$
    (E) $3x^2 \ln\left(x^6 + 1\right)$

13. (calculator not allowed)

$\int_2^3 \frac{x}{x^2+1}\,dx$

(A) $\frac{1}{2}\ln\frac{3}{2}$
(B) $\frac{1}{2}\ln 2$
(C) $\ln 2$
(D) $2\ln 2$
(E) $\frac{1}{2}\ln 5$

14. (calculator allowed)

Let g be the function given by $g(x) = \int_0^x \sin(t^2)\,dt$ for $-1 \le x \le 3$. On which of the following intervals is g decreasing?

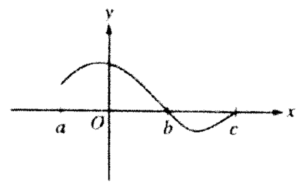
(A) $-1 \le x \le 0$
(B) $0 \le x \le 1.772$
(C) $1.253 \le x \le 2.171$
(D) $1.772 \le x \le 2.507$
(E) $2.802 \le x \le 3$

15. (calculator allowed)

If $0 \le x \le 4$, of the following, which is the greatest value of x such that $\int_0^x (t^2 - 2t)\,dt \ge \int_2^x t\,dt$?

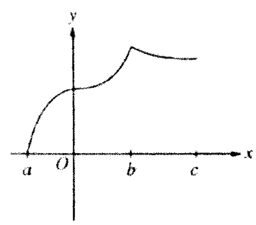
(A) 1.35
(B) 1.38
(C) 1.41
(D) 1.48
(E) 1.59

16. (calculator allowed)



Let $f(x)=\int_a^x h(t)\,dt$, where h has the graph shown above. Which of the following could be the graph of f?

(A)



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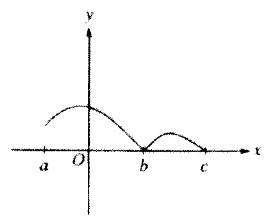

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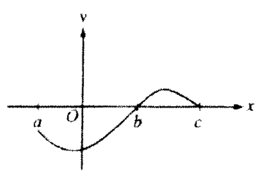

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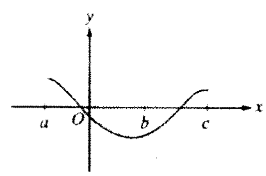
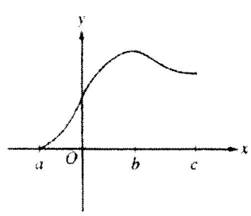


(E)



17. (calculator allowed)

For all values of x, the continuous function f is positive and decreasing. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$. Which of the following could be a table of values for g?

(A)

x	$g(x)$
1	−2
2	0
3	1

(B)

x	$g(x)$
1	−2
2	0
3	3

(C)

x	$g(x)$
1	1
2	0
3	−2

(D)

x	$g(x)$
1	2
2	0
3	−1

(E)

x	$g(x)$
1	3
2	0
3	2

18. (calculator allowed)

The rate of change of the altitude of a hot-air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for $0 \le t \le 8$. Which of the following expressions gives the change in altitude of the balloon during the time the altitude is decreasing?

(A) $\int_{1.572}^{3.514} r(t)\,dt$

(B) $\int_0^8 r(t)\,dt$

(C) $\int_0^{2.667} r(t)\,dt$

(D) $\int_{1.572}^{3.514} r'(t)\,dt$

(E) $\int_0^{2.667} r'(t)\,dt$

19. (calculator allowed)

A pizza, heated to a temperature of 350 degrees Fahrenheit (^{o}F), is taken out of an oven and placed in a $75^{o}F$ room at time $t = 0$ minutes. The temperature of the pizza is changing at a rate of $-110e^{-0.4t}$ degrees Fahrenheit per minute. To the nearest degree, what is the temperature of the pizza at time $t = 5$ minutes?

(A) $112°F$
(B) $119°F$
(C) $147°F$
(D) $238°F$
(E) $335°F$

20. (calculator allowed)

Insects destroyed a crop at the rate of $\dfrac{100e^{-0.1t}}{2-e^{-3t}}$ tons per day, where time t is measured in days. To the nearest ton, how many tons did the insects destroy during the time interval $7 \le t \le 14$?

(A) 125
(B) 100
(C) 88
(D) 50
(E) 12

Free Response

21. (calculator allowed)
There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. ($t = 6$). The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t < 9. \end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

22. (calculator allowed)

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2-t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

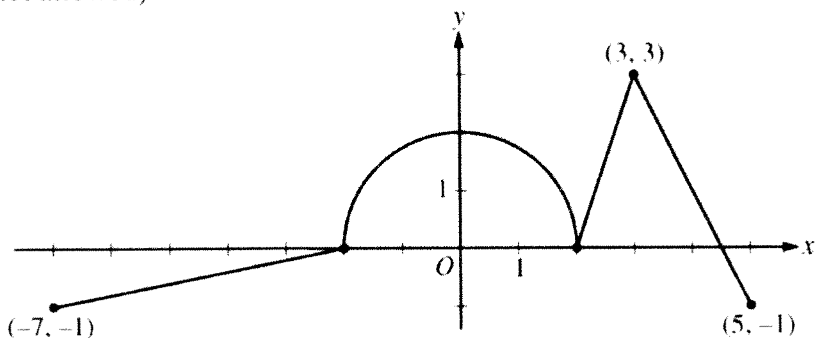
23. (calculator not allowed)

x	2	3	5	8	13
$f(x)$	1	4	−2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

24. (calculator not allowed)



Graph of g'

The function g is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of g, consists of a semicircle and three line segments, as shown in the figure above.

(a) Find $g(3)$ and $g(-2)$.

25. (calculator not allowed)

The functions f and g are given by $f(x) = \int_0^{3x} \sqrt{4 + t^2}\, dt$ and $g(x) = f(\sin x)$.

(a) Find $f'(x)$ and $g'(x)$.

(c) Write, but do not evaluate, an integral expression that represents the maximum value of g on the interval $0 \le x \le \pi$. Justify your answer.

26. (calculator allowed)

At a certain height, a tree trunk has a circular cross section. The radius $R(t)$ of that cross section grows at a rate modeled by the function

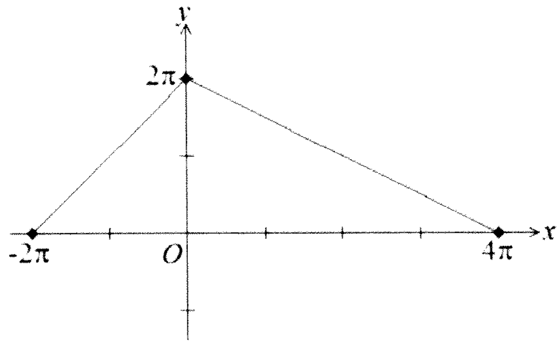
$$\frac{dR}{dt} = \frac{1}{16}\left(3 + \sin(t^2)\right) \text{ centimeters per year}$$

for $0 \le t \le 3$, where time t is measured in years. At time $t = 0$, the radius is 6 centimeters. The area of the cross section at time t is denoted by $A(t)$.

(a) Write an expression, involving an integral, for the radius $R(t)$ for $0 \le t \le 3$. Use your expression to find $R(3)$.

(c) Evaluate $\int_0^3 A'(t)\,dt$. Using appropriate units, interpret the meaning of that integral in terms of cross-sectional area.

27. (calculator not allowed)



Let g be the piecewise-linear defined function on $[-2\pi, 4\pi]$ whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$.

(a) Find $\int_{-2\pi}^{4\pi} f(x)\,dx$. Show the computations that lead to your answer.

(c) Let $h(x) = \int_0^{3x} g(t)\,dt$. Find $h'\left(-\frac{\pi}{3}\right)$.