Area and Volume

Area between two curves

- Sketch the region and determine the points of intersection.
- Draw a small strip either as dx or dy slicing.
- Use the following templates to set up a definite integral:

dx slicing: $A = \int_{left\ x}^{right\ x} \left(y_{top} - y_{bottom}\right) dx$ where y_{top} and y_{bottom} are written in terms of x.

dy slicing: $A = \int_{bottom\ y}^{top\ y} \left(x_{right} - x_{left}\right) dy$ where x_{right} and x_{left} are written in terms of y.

Volume of a Solid with a Known Cross-Section

- Sketch the region and draw a representative rectangle to be used in determining whether setting up with respect to dx or dy.
- Determine the slicing direction then find the volume of the slice which will be the area of the "face" times the "thickness".
- Determine the total volume by summing up the slices using a definite integral.
- Use the following templates to set up a definite integral.

dx slicing: $V = \int_{left\ x}^{right\ x} A(x)\, dx$ where $A(x)$ is the area of the face written in terms of x.

dy slicing: $V = \int_{bottom\ y}^{top\ y} A(y)\, dy$ *where* $A(y)$ is the area of the face written in terms of y.

Volume of a Solid of Revolution

- Sketch the region to be revolved and a representative rectangle whose width can be used to determine whether integrating with dx or dy.
- Set up a definite integral after determining whether the slicing uses dx or dy so that the slicing is perpendicular to the axis of revolution.
- Identify the outside radius and the inside radius and use the appropriate template from below:

- *dx slicing:* $V = \pi \int_{left\ x}^{right\ x} \left(\left(outside\ radius\right)^2 - \left(inside\ radius\right)^2\right) dx$

  where the outside and inside radii are written in terms of x.

  dy slicing: $V = \pi \int_{bottom\ y}^{top\ y} \left(\left(outside\ radius\right)^2 - \left(inside\ radius\right)^2\right) dy$

  where the outside and inside radii are written in terms of y.

Multiple Choice

1. (calculator not allowed)
The region enclosed by the x-axis, the line $x = 3$, and the curve $y = \sqrt{x}$ is rotated about the x-axis. What is the volume of the solid generated?

(A) 3π
(B) $2\sqrt{3}\pi$
(C) $\frac{9}{2}\pi$
(D) 9π
(E) $\frac{36\sqrt{3}}{5}\pi$

2. (calculator not allowed)
The area of the region bounded by the lines $x = 0$, $x = 2$, and $y = 0$ and the curve $y = e^{\frac{x}{2}}$ is

(A) $\frac{e-1}{2}$
(B) $e-1$
(C) $2(e-1)$
(D) $2e-1$
(E) $2e$

3. (calculator not allowed)

The region bounded by the x-axis and the part of the graph of $y = \cos x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is separated into two regions by the line $x = k$. If the area of the region for $-\frac{\pi}{2} \le x \le k$ is three times the area of the region for $k \le x \le \frac{\pi}{2}$, then $k =$

(A) $\arcsin\left(\frac{1}{4}\right)$

(B) $\arcsin\left(\frac{1}{3}\right)$

(C) $\frac{\pi}{6}$

(D) $\frac{\pi}{4}$

(E) $\frac{\pi}{3}$

4. (calculator allowed)

The base of a solid is the region in the first quadrant bounded by the y-axis, the graph of $y = \tan^{-1} x$, the horizontal line $y = 3$, and the vertical line $x = 1$. For this solid, each cross section perpendicular to the *x-axis* is a square. What is the volume of the solid?

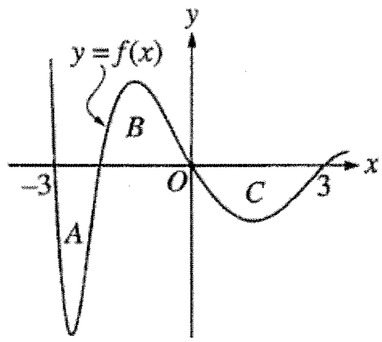
(A) 2.561
(B) 6.612
(C) 8.046
(D) 8.755
(E) 20.773

5. (calculator allowed)

What is the area enclosed by the curves $y = x^3 - 8x^2 + 18x - 5$ and $y = x + 5$?

(A) 10.667
(B) 11.833
(C) 14.583
(D) 21.333
(E) 32

6. (calculator allowed)

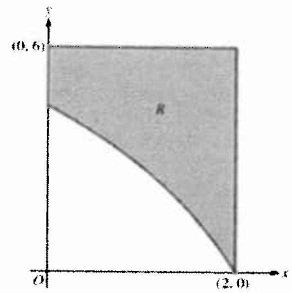


The regions A, B, and C in the figure above are bounded by the graph of the function f and the x – axis. If the area of each region is 2, what is the value of $\int_{-3}^{3}(f(x)+1)\,dx$?

(A) −2
(B) −1
(C) 4
(D) 7
(E) 12

Free Response

7. (calculator allowed)



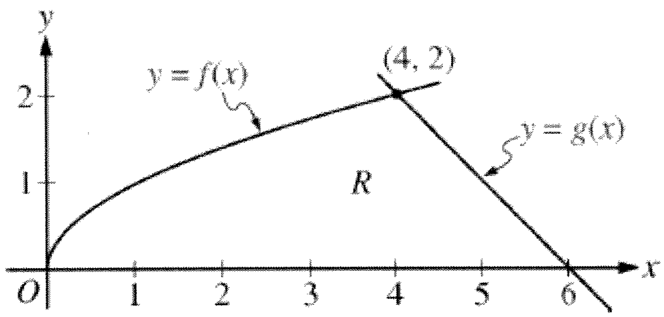
In the figure above, R, is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

8. (calculator not allowed)



The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.

(a) Find the area of R.

(b) The region R is the base of a solid. For each y, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(c) There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g. Find the coordinates of point.

9. (calculator allowed)

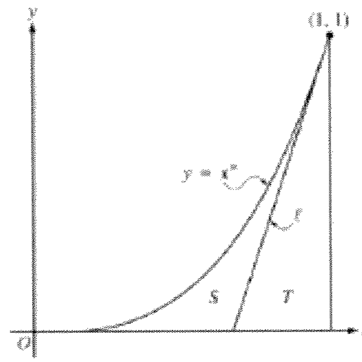
Let R be the region in the first and second quadrants bounded above by the graph of $y = \frac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of R.

(b) Find the volume generated when R is rotated about the x-axis.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

10. (calculator not allowed)

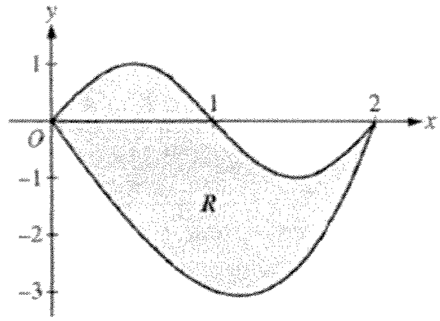


Let ℓ be the line tangent to the graph of $y = x^n$ at the point (1, 1), where $n > 1$, as shown above.

(a) Find $\int_0^1 x^n\,dx$ in terms of n.

(b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\dfrac{1}{2n}$.

(c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.



11. (calculator allowed) (2008 AB1/BC1)
Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.