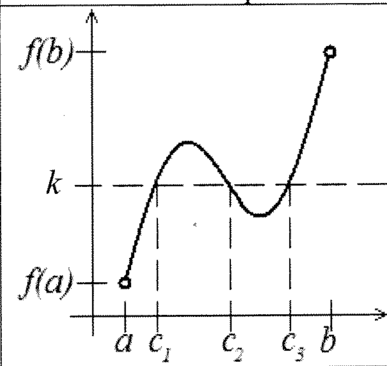
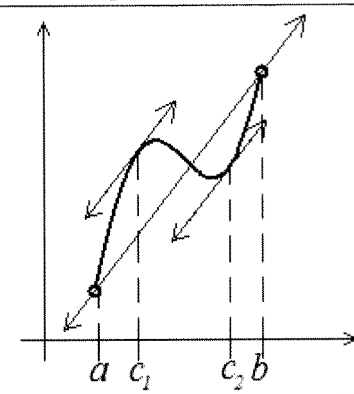
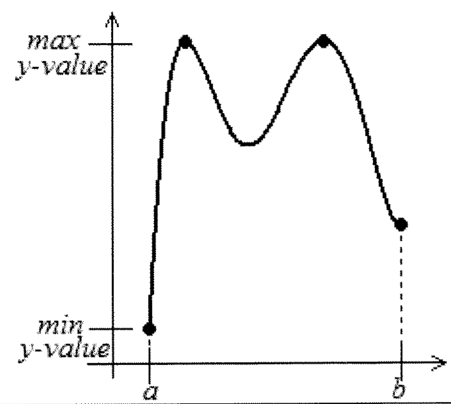
Theorems

Students should be able to apply and have a geometric understanding of the following:

- Intermediate Value Theorem
- Mean Value Theorem for derivatives
- Extreme Value Theorem
- Average Value Theorem for integrals

Name	Formal Statement	Restatement	Graph	Notes
IVT	If $f(x)$ is continuous on a closed interval $[a,b]$ and $f(a) \neq f(b)$, then for every value m between $f(a)$ and $f(b)$ there exists at least one value c in (a,b) such that $f(c)=k$.	On a continuous function, you will hit every y-value between two given y-values at least once.		Examples: height, temperature Non-examples: bank account
MVT	If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on (a,b), then there must exist at least one value c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$	If conditions are met (very important!) there is at least one point where the slope of the tangent line equals the slope of the secant line.		Can also be used to guarantee a point when instantaneous rate of change = average rate of change.
EVT	A continuous function $f(x)$ on a closed interval $[a,b]$ attains both an absolute maximum $f(c) \geq f(x)$ for all x in the interval and an absolute minimum $f(c) \leq f(x)$ for all x in the interval	Every continuous function on a closed interval has a highest y-value and a lowest y-value.		Max/Min values can occur at endpoints or critical points. The max/min value can occur at more than one x-value.

Multiple Choice

1. (calculator not allowed)
   If f is continuous for $a \le x \le b$ and differentiable for $a < x < b$, which of the following could be false?

   (A) $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some c such that $a < c < b$.
   (B) $f'(c) = 0$ for some c such that $a < c < b$.
   (C) f has a minimum value on $a \le x \le b$.
   (D) f has a maximum value on $a \le x \le b$.
   (E) $\int_a^b f(x)\,dx$ exists.

2. (calculator not allowed)
   Let f be a polynomial function with degree greater than 2. If $a \ne b$ and $f(a) = f(b) = 1$, Which of the following must be true for at least one value of x between a and b?
   I. $f(x) = 0$
   II. $f'(x) = 0$
   III. $f''(x) = 0$

   (A) None
   (B) I only
   (C) II only
   (D) I and II only
   (E) I, II, and III

3. (calculator not allowed)
   Let f be the function given by $f(x) = x^3 - 3x^2$. What are all values of c that satisfy the conclusion of the Mean Value Theorem of differential calculus on the closed interval $[0, 3]$?

   (A) 0 only
   (B) 2 only
   (C) 3 only
   (D) 0 and 3
   (E) 2 and 3

4. (calculator not allowed)

x	0	1	2
$f(x)$	1	k	2

The function f is continuous on the closed interval [0, 2] and has values that are given in the table above. The equation $f(x)=\frac{1}{2}$ must have at least two solutions in the interval [0, 2] if $k =$

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) 2
(E) 3

5. (calculator not allowed)
Let g be a continuous function on the closed interval [0, 1]. Let $g(0)=1$ and $g(1)=0$. Which of the following is NOT necessarily true?

(A) There exists a number h in [0, 1] such that $g(h) \geq g(x)$ for all x in [0, 1].

(B) For all a and b in [0, 1], if $a = b$, then $g(a)=g(b)$.

(C) There exists a number h in [0, 1] such that $g(h)=\frac{1}{2}$.

(D) There exists a number h in [0, 1] such that $g(h)=\frac{3}{2}$.

(E) For all h in the open interval (0, 1), $\lim_{x \to h} g(x)=g(h)$.

6. (calculator not allowed)
If $f(x)=\sin\left(\frac{x}{2}\right)$, then there exists a number c in the interval $\frac{\pi}{2}<x<\frac{3\pi}{2}$ that satisfies the conclusion of the Mean Value Theorem. Which of the following could be c?

(A) $\frac{2\pi}{3}$
(B) $\frac{3\pi}{4}$
(C) $\frac{5\pi}{6}$
(D) π
(E) $\frac{3\pi}{2}$

7. (calculator not allowed)
If f is continuous on the closed interval $[a,b]$, then there exists c such that $a<c<b$ and $\int_a^b f(x)\,dx =$

(A) $\dfrac{f(c)}{b-a}$
(B) $\dfrac{f(b)-f(a)}{b-a}$
(C) $f(b)-f(a)$
(D) $f'(c)(b-a)$
(E) $f(c)(b-a)$

8. (calculator not allowed)
The Mean Value Theorem guarantees the existence of a special point on the graph of $y=\sqrt{x}$ between (0, 0) and (4, 2). What are the coordinates of this point?

(A) (2, 1)
(B) (1, 1)
(C) $\left(2,\sqrt{2}\right)$
(D) $\left(\dfrac{1}{2},\dfrac{1}{\sqrt{2}}\right)$
(E) None of the above

9. (calculator allowed)
Let f be a function that is differentiable on the open interval (1, 10). If $f(2)=-5$, $f(5)=5$, and $f(9)=-5$, which of the following must be true?

I. f has at least 2 zeros.

II. The graph of f has at least one horizontal tangent.

III. For some c, $2<c<5$, $f(c)=3$.

(A) None
(B) I only
(C) I and II only
(D) I and III only
(E) I, II and III

10. (calculator allowed)
The function f is continuous for $-2 \le x \le 2$ and $f(-2) = f(2) = 0$. If there is no c, where $-2 < c < 2$, for which $f'(c) = 0$, which of the following statements must be true?

(A) For $-2 < k < 2, f'(k) > 0$.
(B) For $-2 < k < 2, f'(k) < 0$.
(C) For $-2 < k < 2, f'(k)$ exists.
(D) For $-2 < k < 2, f'(k)$ exists, but f' is not continuous.
(E) For some k, where $-2 < k < 2, f'(k)$ does not exist.

11. (calculator allowed)
The function f is continuous for $-2 \le x \le 1$ and differentiable for $-2 < x < 1$. If $f(-2) = -5$ and $f(1) = 4$, which of the following statements could be false?

(A) There exists c, where $-2 < c < 1$, such that $f(c) = 0$.
(B) There exists c, where $-2 < c < 1$, such that $f'(c) = 0$.
(C) There exists c, where $-2 < c < 1$, such that $f(c) = 3$.
(D) There exists c, where $-2 < c < 1$, such that $f'(c) = 3$.
(E) There exists c, where $-2 \le c \le 1$ such that $f(c) \ge f(x)$ for all x on the closed interval $-2 \le x \le 1$.

12. (calculator allowed)
Let f be a continuous function on the closed interval [–3, 6]. If $f(-3) = -1$ and $f(6) = 3$, then the Intermediate Value Theorem guarantees that

(A) $f(0) = 0$
(B) $f'(c) = \frac{4}{9}$ for at least one c between –3 and 6
(C) $-1 \le f(x) \le 3$ for all x between –3 and 6
(D) $f(c) = 1$ for at least one c between –3 and 6
(E) $f(c) = 0$ for at least one c between –1 and 3

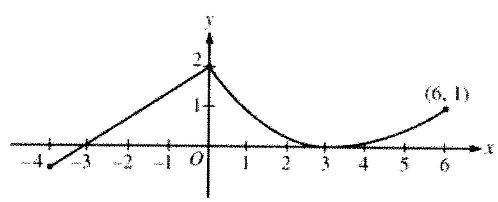
13. (calculator allowed)

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24-hour period.

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

14. (calculator allowed)



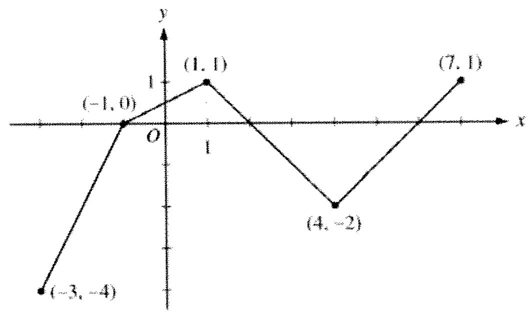
Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(b) For how many values of a, $-4 \le a < 6$, is the average rate of change of f on the interval $[a, 6]$ equal to zero? Give a reason for your answer.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

15. (calculator not allowed)



Graph of g'

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

16. (calculator allowed)

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	−4	4	2
4	−1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

17. (calculator not allowed)

Let f be a twice-differentiable function such that $f(2) = 5$ and $f(5) = 2$. Let g be the function given by $g(x) = f(f(x))$.

(a) Explain why there must be a value c for $2 < c < 5$ such that $f'(c) = -1$.

(b) Show that $g'(2) = g'(5)$. Use this result to explain why there must be a value k for $2 < k < 5$ such that $g''(k) = 0$.

(d) Let $h(x) = f(x) - x$. Explain why there must be a value r for $2 < r < 5$ such that $h(r) = 0$.

18. (calculator not allowed)

x	-1.5	-1.0	-0.5	0	0.5	1.0	1.5
$f(x)$	-1	-4	-6	-7	-6	-4	-1
$f'(x)$	-7	-5	-3	0	3	5	7

Let f be a function that is differentiable for all real numbers. The table above gives the values of f and its derivative f' for selected points x in the closed interval $-1.5 \le x \le 1.5$. The second derivative of f has the property that $f''(x) > 0$ for $-1.5 \le x \le 1.5$.

(c) Find a positive real number r having the property that there must exist a value c with $0 < c < 0.5$ and $f''(c) = r$. Give a reason for your answer.