

Optimization

Relative or Local Extrema – highest or lowest point in the neighborhood

- First derivative test
 - Candidates – critical numbers (x -values that make f' zero or undefined where f is defined)
 - Test – (1) set up an f' number line; label with candidates
 (2) test each section to see if f' is positive or negative
 (3) relative maximum occurs when f' changes from + to –
 relative minimum occurs when f' changes from – to +
- Second derivative test
 - Candidates – critical numbers (x -values that make f' zero or undefined where f is defined)
 - Test – (1) substitute each critical number into the second derivative
 (2) $f'' > 0$, relative minimum
 $f'' < 0$, relative maximum
 (3) $f'' = 0$, the test fails

Absolute or Global Extrema – highest or lowest point in the domain

- Absolute Extrema Test
 - Candidates – critical numbers and endpoints of the domain
 - Test – (1) find the y -values for each candidate
 (2) the absolute maximum value is the largest y -value,
 the absolute minimum value is the smallest y -value

Students need to be able to:

- Locate a function's relative (local) and absolute (global) extrema using the first derivative test, second derivative test or closed interval test (also known as candidates test).
- Reason from a graph without finding an explicit rule that represents the graph.
- Write justifications and explanations.
 - Must be written in sentence form.
 - Avoid using the pronoun "it" when justifying extrema.
 - Use "the function," "the derivative," or "the second derivative" instead of "the graph" or "the slope" in explanations.

Multiple Choice

1. (calculator not allowed)

The function defined by $f(x) = x^3 - 3x^2$ for all real numbers x has a relative maximum at $x =$

- (A) -2
- (B) 0
- (C) 1
- (D) 2
- (E) 4

2. (calculator not allowed)

If $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 5$ and the domain is the set of all x such that $0 \leq x \leq 9$, then the absolute maximum value of the function f occurs when x is

- (A) 0
- (B) 2
- (C) 4
- (D) 6
- (E) 9

3. (calculator not allowed)

The volume of a cylindrical tin can with a top and a bottom is to be 16π cubic inches. If a minimum amount of tin is to be used to construct the can, what must be the height, in inches, of the can?

- (A) $2\sqrt[3]{2}$
- (B) $2\sqrt{2}$
- (C) $2\sqrt[3]{4}$
- (D) 4
- (E) 8

4. (calculator not allowed)

For what value of x does the function $f(x) = (x-2)(x-3)^2$ have a relative maximum?

- (A) -3
- (B) $-\frac{7}{3}$
- (C) $-\frac{5}{2}$
- (D) $\frac{7}{3}$
- (E) $\frac{5}{2}$

5. (calculator not allowed)

What is the minimum value of $f(x) = x \ln x$?

- (A) $-e$
- (B) -1
- (C) $-\frac{1}{e}$
- (D) 0
- (E) $f(x)$ has no minimum value.

6. (calculator not allowed)

The derivative of f is $x^4(x-2)(x+3)$. At how many points will the graph of f have a relative maximum?

- (A) None
- (B) One
- (C) Two
- (D) Three
- (E) Four

7. (calculator not allowed)

Consider all right circular cylinders for which the sum of the height and the circumference is 30 centimeters. What is the radius of the one with maximum volume?

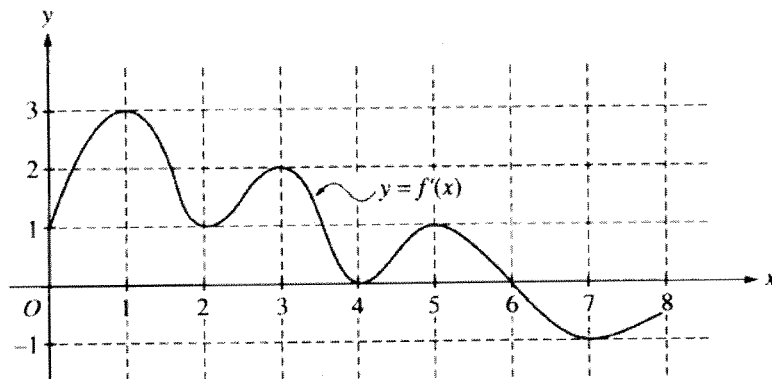
- (A) 3 cm
- (B) 10 cm
- (C) 20 cm
- (D) $\frac{30}{\pi^2}$ cm
- (E) $\frac{10}{\pi}$ cm

8. (calculator not allowed)

The point on the curve $x^2 + 2y = 0$ that is nearest the point $\left(0, -\frac{1}{2}\right)$ occurs where y is

- (A) $\frac{1}{2}$
- (B) 0
- (C) $-\frac{1}{2}$
- (D) -1
- (E) none

9. (calculator not allowed)



The function f is defined on the closed interval $[0, 8]$. The graph of its derivative f' is shown above. At what value of x does the absolute minimum of f occur?

- (A) 0
- (B) 2
- (C) 4
- (D) 6
- (E) 8

10. (calculator not allowed)

What is the area of the largest rectangle that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$?

- (A) $6\sqrt{2}$
- (B) 12
- (C) 24
- (D) $24\sqrt{2}$
- (E) 36

11. (calculator allowed)

If $y = 2x - 8$, what is the minimum value of the product of xy ?

- (A) -16
- (B) -8
- (C) -4
- (D) 0
- (E) 2

Free Response Questions

12. (calculator not allowed)

Let f be the function given by $f(x) = \frac{\ln(x)}{x}$ for all $x > 0$. The derivative of f is given by

$$f'(x) = \frac{1 - \ln(x)}{x^2}.$$

(b) Find the x -coordinate of the critical point of f . Determine whether this point is a relative minimum, a relative maximum, or neither for the function f . Justify your answer.

13. (calculator allowed)

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \leq t \leq 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

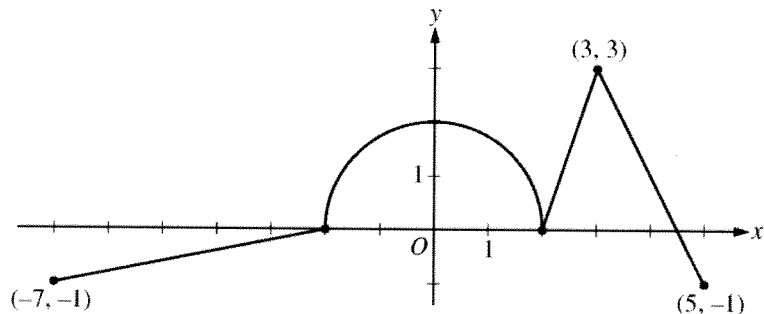
(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

14. (calculator allowed)

(d) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P , where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \leq t \leq 12$.

According to the model, at what time were the entries being processed most quickly? Justify your answer.

15. (calculator not allowed)

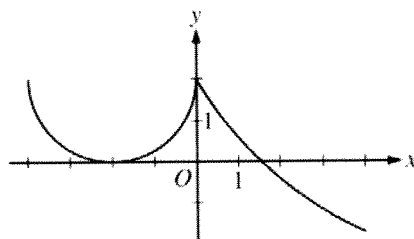


Graph of g'

The function g is defined and differentiable on the closed interval $[-7, 5]$ and satisfies $g(0) = 5$. The graph of $y = g'(x)$, the derivative of g , consists of a semicircle and three line segments, as shown in the figure above.

(c) The function h is defined by $h(x) = g(x) - \frac{1}{2}x^2$. Find the x -coordinate of each critical point of h , where $-7 < x < 5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.

16. (calculator not allowed)



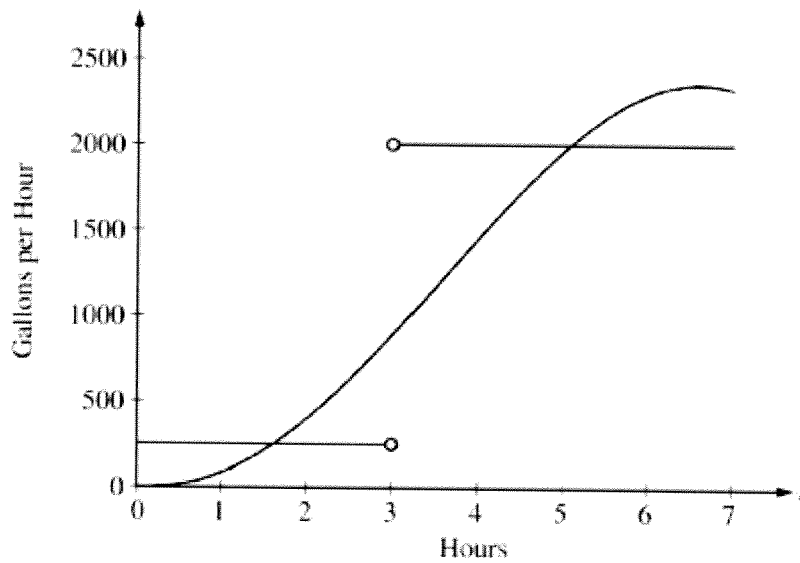
Graph of f'

The derivative of a function f is defined by $f'(x) = \begin{cases} g(x) & \text{for } -4 \leq x \leq 0 \\ 5e^{\frac{-x}{3}} - 3 & \text{for } 0 < x \leq 4 \end{cases}$.

The graph of the continuous function f' , shown in the figure above, has x -intercepts at $x = -2$ and $x = 3 \ln\left(\frac{5}{3}\right)$. The graph g on $-4 \leq x \leq 0$ is a semicircle, and $f(0) = 5$.

(c) For $-4 \leq x \leq 4$, find the value of x at which f has an absolute maximum. Justify your answer.

17. (calculator allowed)



The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \leq t \leq 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin(\sqrt{t})$ gallons per hour for $0 \leq t \leq 7$.

(ii) The rate at which water leaves the tank is

$$g(x) = \begin{cases} 250 & \text{for } 0 \leq t < 3 \\ 2000 & \text{for } 3 < t \leq 7 \end{cases} \text{ gallons per hour.}$$

The graphs of f and g , which intersect at $t = 1.617$ and $t = 5.076$ are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(c) For $0 \leq t \leq 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

