



## Limits, Continuity, and Differentiability

### Continuity

A function is continuous on an interval if it is continuous at every point of the interval. Intuitively, a function is continuous if its graph can be drawn without ever needing to pick up the pencil. This means that the graph of  $y = f(x)$  has no “holes”, no “jumps” and no vertical asymptotes at  $x = a$ . When answering free response questions on the AP exam, the formal definition of continuity is required. To earn all of the points on the free response question scoring rubric, all three of the following criteria need to be met, with work shown:

A function is continuous at a point  $x = a$  if and only if:

1.  $f(a)$  exists
2.  $\lim_{x \rightarrow a} f(x)$  exists
3.  $\lim_{x \rightarrow a} f(x) = f(a)$  (i.e., the limit equals the function value)

### Continuity and Differentiability

Differentiability implies continuity (but not necessarily vice versa) If a function is differentiable at a point (at every point on an interval), then it is continuous at that point (on that interval). The converse is not always true: continuous functions may not be differentiable.

### Quick Check for Understanding:

1. Sketch a function with the property that  $f(a)$  exists but  $\lim_{x \rightarrow a} f(x)$  does not exist.
2. Sketch a function with the property that  $\lim_{x \rightarrow a} f(x)$  exists but  $f(a)$  does not exist.
3. Sketch a function with the property that  $f(a)$  exists and  $\lim_{x \rightarrow a} f(x)$  exists but  $\lim_{x \rightarrow a} f(x) \neq f(a)$ .

Students should be able to:

- Determine limits from a graph
- Know the relationship between limits and asymptotes (i.e., limits that become infinite at a finite value or finite limits at infinity)
- Compute limits algebraically
- Discuss continuity algebraically and graphically and know its relation to limit.
- Discuss differentiability algebraically and graphically and know its relation to limits and continuity
- Recognize the limit definition of derivative and be able to identify the function involved and the point at which the derivative is evaluated. For example, since

$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ , recognize that  $\lim_{h \rightarrow 0} \frac{\cos(\pi+h) - \cos(\pi)}{h}$  is simply the derivative of  $\cos(x)$  at  $x = \pi$ .

- L'Hôpital's Rule (BC only)
- Limits associated with logistic equations (BC only)

Multiple Choice

1. (calculator not allowed) (1985AB5)

$$\lim_{n \rightarrow \infty} \frac{4n^2}{n^2 + 10000n} \text{ is}$$

- (A) 0
- (B)  $\frac{1}{2500}$
- (C) 1
- (D) 4
- (E) nonexistent

2. (calculator not allowed)

$$\lim_{x \rightarrow \infty} \frac{(2x-1)(3-x)}{(x-1)(x+3)} \text{ is}$$

- (A) -3
- (B) -2
- (C) 2
- (D) 3
- (E) nonexistent

3. (calculator not allowed)

$$\text{What is } \lim_{h \rightarrow 0} \frac{8\left(\frac{1}{2} + h\right)^8 - 8\left(\frac{1}{2}\right)^8}{h} ?$$

- (A) 0
- (B)  $\frac{1}{2}$
- (C) 1
- (D) The limit does not exist.
- (E) It cannot be determined from the information given.

4. (calculator not allowed)

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}} \text{ is}$$

- (A) 0
- (B)  $\frac{1}{\sqrt{2}}$
- (C)  $\frac{\pi}{4}$
- (D) 1
- (E) nonexistent

5. (calculator not allowed)

The  $\lim_{h \rightarrow 0} \frac{\tan 3(x+h) - \tan 3x}{h}$  is

- (A) 0
- (B)  $3 \sec^2(3x)$
- (C)  $\sec^2(3x)$
- (D)  $3 \cot(3x)$
- (E) nonexistent

6. (calculator not allowed)

If  $f(x) = 2x^2 + 1$ , then  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x^2}$  is

- (A) 0
- (B) 1
- (C) 2
- (D) 4
- (E) nonexistent

7. (calculator not allowed)

If  $f'(x) = \cos x$  and  $g'(x) = 1$  for all  $x$ , and if  $f(0) = g(0) = 0$ , then  $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)}$  is

- (A)  $\frac{\pi}{2}$
- (B) 1
- (C) 0
- (D) -1
- (E) nonexistent

8. (calculator not allowed)

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{2 \sin^2 \theta} \text{ is}$$

- (A) 0
- (B)  $\frac{1}{8}$
- (C)  $\frac{1}{4}$
- (D) 1
- (E) nonexistent

9. (calculator not allowed)

If  $\lim_{x \rightarrow a} f(x) = L$  where  $L$  is a real number, which of the following must be true?

- (A)  $f'(a)$  exists.
- (B)  $f(x)$  is continuous at  $x = a$ .
- (C)  $f(x)$  is defined at  $x = a$ .
- (D)  $f(a) = L$
- (E) None of the above

10. (calculator not allowed)

For  $x \geq 0$ , the horizontal line  $y = 2$  is an asymptote for the graph of the function  $f$ . Which of the following statements must be true?

- (A)  $f(0) = 2$
- (B)  $f(x) \neq 2$  for all  $x \geq 0$
- (C)  $f(2)$  is undefined.
- (D)  $\lim_{x \rightarrow 2} f(x) = \infty$
- (E)  $\lim_{x \rightarrow \infty} f(x) = 2$

11. (calculator not allowed)

If the graph of  $y = \frac{ax+b}{x+c}$  has a horizontal asymptote at  $y = 2$  and a vertical asymptote at  $x = -3$ , then  $a+c =$

- (A)  $-5$
- (B)  $-1$
- (C)  $0$
- (D)  $1$
- (E)  $5$

12. (calculator not allowed)

At  $x = 3$ , the function given by  $f(x) = \begin{cases} x^2, & x < 3 \\ 6x-9, & x \geq 3 \end{cases}$  is

- (A) undefined.
- (B) continuous but not differentiable.
- (C) differentiable but not continuous.
- (D) neither continuous nor differentiable.
- (E) both continuous and differentiable.

13. (calculator not allowed)

$$f(x) = \begin{cases} x+2 & \text{if } x \leq 3 \\ 4x-7 & \text{if } x > 3 \end{cases}$$

Let  $f$  be the function given above. Which of the following statements are true about  $f$ ?

- I.  $\lim_{x \rightarrow 3} f(x)$  exists.
- II.  $f$  is continuous at  $x = 3$ .
- III.  $f$  is differentiable at  $x = 3$ .

- (A) None
- (B) I only
- (C) II only
- (D) I and II only
- (E) I, II and III

14. (calculator not allowed)

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Let  $f$  be the function defined above. Which of the following statements about  $f$  are true?

- I.  $f$  has a limit at  $x = 2$ .
- II.  $f$  is continuous at  $x = 2$ .
- III.  $f$  is differentiable at  $x = 2$ .

- (A) I only
- (B) II only
- (C) III only
- (D) I and II only
- (E) I, II, and III

15. (calculator not allowed)

Let  $f$  be defined by the following.

$$f(x) = \begin{cases} \sin x, & x < 0 \\ x^2, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x \leq 2 \\ x - 3, & x \geq 2 \end{cases}$$

For what values of  $x$  is  $f$  NOT continuous?

- (A) 0 only
- (B) 1 only
- (C) 2 only
- (D) 0 and 2 only
- (E) 0, 1, and 2

16. (calculator not allowed)

If  $f(x) = 2 + |x - 3|$  for all  $x$ , then the value of the derivative  $f'(x)$  at  $x = 3$  is

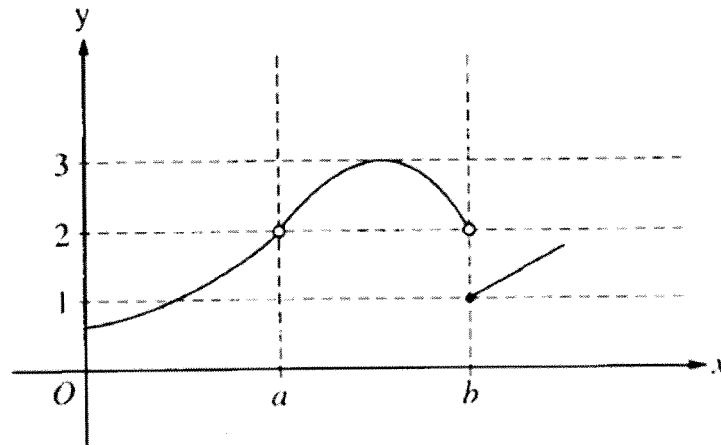
- (A)  $-1$
- (B)  $0$
- (C)  $1$
- (D)  $2$
- (E) nonexistent

17. (calculator not allowed)

If the function  $f$  is continuous for all real numbers and if  $f(x) = \frac{x^2 - 4}{x + 2}$  when  $x \neq -2$ ,  
then  $f(-2) =$

- (A) -4
- (B) -2
- (C) -1
- (D) 0
- (E) 2

18. (calculator not allowed)

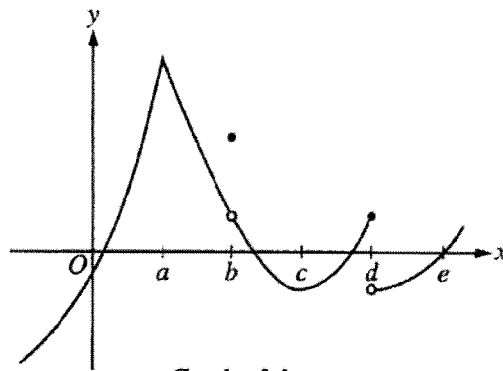


The graph of the function  $f$  is shown in the figure above. Which of the following statements about  $f$  is true?

- (A)  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow b} f(x)$
- (B)  $\lim_{x \rightarrow a} f(x) = 2$
- (C)  $\lim_{x \rightarrow b} f(x) = 2$
- (D)  $\lim_{x \rightarrow b} f(x) = 1$
- (E)  $\lim_{x \rightarrow a} f(x)$  does not exist.



19. (calculator not allowed)

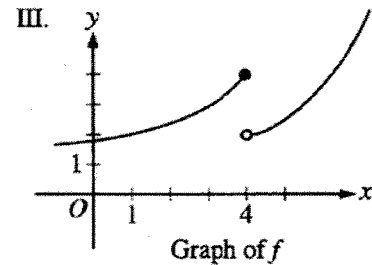
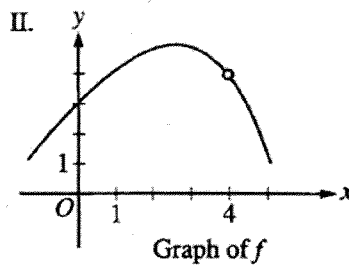
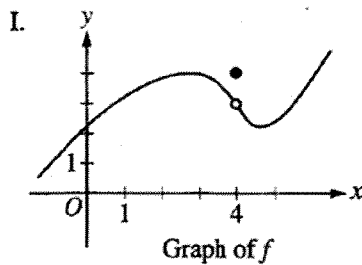


Graph of  $f$

The graph of a function  $f$  is shown above. At which value of  $x$  is  $f$  continuous, but not differentiable?

- (A)  $a$
- (B)  $b$
- (C)  $c$
- (D)  $d$
- (E)  $e$

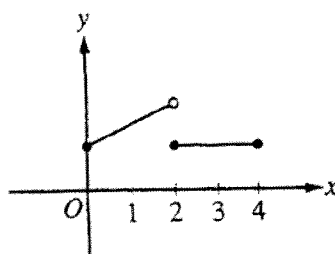
20. (calculator allowed)



For which of the following does  $\lim_{x \rightarrow 4} f(x)$  exist?

- (A) I only
- (B) II only
- (C) III only
- (D) I and II only
- (E) I and III only

21. (calculator allowed)



Graph of  $f$

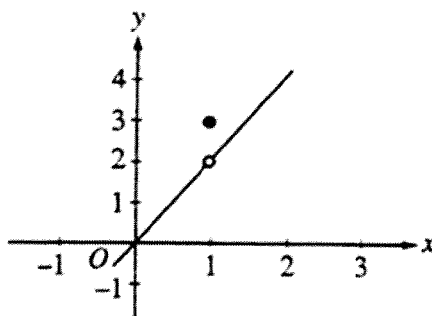
The figure above shows the graph of a function  $f$  with domain  $0 \leq x \leq 4$ . Which of the following statements are true?

I.  $\lim_{x \rightarrow 2^-} f(x)$  exists

II.  $\lim_{x \rightarrow 2^+} f(x)$  exists

III.  $\lim_{x \rightarrow 2} f(x)$  exists

(A) I only    (B) II only    (C) I and II only    (D) I and III only    (E) I, II, and III



Graph of  $f$

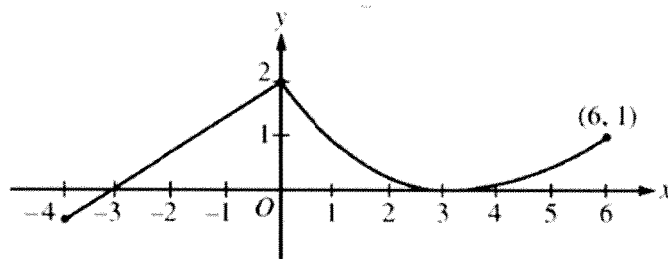
22. (calculator allowed)

The graph of the function  $f$  is shown in the figure above. The value of  $\lim_{x \rightarrow 1} \sin(f(x))$  is

- (A) 0.909
- (B) 0.841
- (C) 0.141
- (D) -0.416
- (E) nonexistent

Free Response

23. (calculator allowed)



Graph of  $f$

A continuous function  $f$  is defined on the closed interval  $-4 \leq x \leq 6$ . The graph of  $f$  consists of a line segment and a curve that is tangent to the  $x$ -axis at  $x = 3$ , as shown in the figure above. On the interval  $0 < x < 6$ , the function  $f$  is twice differentiable, with  $f''(x) > 0$ .

- (a) Is  $f$  differentiable at  $x = 0$ ? Use the definition of the derivative with one-sided limits to justify your answer.
  
- (b) For how many values of  $a$ ,  $-4 \leq a < 6$ , is the average rate of change of  $f$  on the interval  $[a, 6]$  equal to 0? Give a reason for your answer.
  
- (c) Is there a value of  $a$ ,  $-4 \leq a < 6$ , for which the Mean Value Theorem, applied to the interval  $[a, 6]$ , guarantees a value  $c$ ,  $a < c < 6$ , at which  $f'(c) = \frac{1}{3}$ ? Justify your answer.

24. (calculator not allowed)

Let  $f$  be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \leq x \leq 3 \\ 5-x & \text{for } 3 < x \leq 5. \end{cases}$$

(a) Is  $f$  continuous at  $x = 3$  ? Explain why or why not.

(c) Suppose the function  $g$  is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \leq x \leq 3 \\ mx+2 & \text{for } 3 < x \leq 5, \end{cases}$$

where  $k$  and  $m$  are constants. If  $g$  is differentiable at  $x = 3$ , what are the values of  $k$  and  $m$ ?

25. (calculator not allowed)

Let  $f$  be a function defined by  $f(x) = \begin{cases} 1 - 2 \sin x, & x \leq 0 \\ e^{-4x}, & x > 0. \end{cases}$

(a) Show that  $f$  is continuous at  $x = 0$ .

(b) For  $x \neq 0$ , express  $f'(x)$  as a piecewise-defined function. Find the value of  $x$  for which  $f'(x) = -3$ .

Grade the student responses for question 25 part (a). Determine if the response should receive full credit (2 points).

NO CALCULATOR ALLOWED

6A,

Work for problem 6(a)

To be continuous

$$i) f(0) = 1$$

$$ii) \lim_{x \rightarrow 0^-} 1 - 2 \sin x = \lim_{x \rightarrow 0^+} e^{-4x}$$

$$\therefore \lim_{x \rightarrow 0} f = 1 \quad \because \lim_{x \rightarrow 0^-} f = \lim_{x \rightarrow 0^+} f$$

$$iii) f(0) = \lim_{x \rightarrow 0} f = 1$$

$\therefore f$  is continuous for all values of  $x$ .

NO CALCULATOR ALLOWED

6B,

Work for problem 6(a)

$$1 - 2 \sin x = e^{-4x}$$

$$e^{-4(0)} = 1 \text{ and } 1 - 2 \sin(0) = 1 - 0 = 1$$

They're both = 1 @  $x=0$ , therefore they're continuous.