Graphical Relationships Among f, f', and f''

The relationship between the graph of a function and its first and second derivatives frequently appears on the AP exams. It will appear on both multiple choice and the free response section, often with the graph of $y = f'(x)$ given (read the titles on the graphs very carefully).

Free response questions on this topic often require students to justify their answers. Students may use number lines to analyze the characteristics of the function; however, the justification must be written in sentence form. This justification should avoid using a pronoun such as "it" to describe the function and should make use of calculus involving the first or second derivative test or theorems.

Increase/Decrease

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on the interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on the interval.

Relative or Local Extrema – highest or lowest point in the neighborhood

- First derivative test
  - Candidates – critical numbers (x-values that make f' zero or undefined where f is defined)
  - Test – (1) set up an f' number line; label with candidates
    (2) test each section to see if f' is positive or negative
    (3) relative maximum occurs when f' changes from + to –
    relative minimum occurs when f' changes from – to +
- Second derivative test
  - Candidates – critical numbers (x-values that make f' zero or undefined where f is defined)
  - Test – (1) substitute each critical number into the second derivative
    (2) $f'' > 0$, relative minimum
    $f'' < 0$, relative maximum
    (3) $f'' = 0$, the test fails

Absolute or Global Extrema – highest or lowest point in the domain

- Absolute Extrema Test
  - Candidates – critical numbers and endpoints of the domain
  - Test – (1) find the y-values for each candidate
    (2) the absolute maximum value is the largest y-value,
    the absolute minimum value is the smallest y-value

Concavity

- If $f''(x) > 0$ (or $f'(x)$ is increasing) on an interval, then $f(x)$ is concave up on that interval.
- If $f''(x) < 0$ (or $f'(x)$ is decreasing) on an interval, then $f(x)$ is concave down on that interval.

Point of inflection – point where the concavity changes

- Determining points of inflection using the first derivative
  - The graph of f has a point of inflection where f' has a maximum or minimum
- Determining points of inflection using the second derivative
  - Candidates – x-values for which f'' is zero or undefined where f is defined
  - Test – (1) set up an f'' number line; label with candidates
    (2) test each section to see if f'' is positive or negative
    (3) any change in the sign of f'' indicates a point of inflection

<u>Justifications for Increasing/Decreasing Intervals of a function</u>

<u>Remember</u>: $f'(x)$ determines whether a function is increasing or decreasing, so always use the sign of $f'(x)$ when determining and justifying whether a function $f(x)$ is increasing or decreasing.

Situation	Explanation
$f(x)$ is increasing on the interval (a, b)	$f(x)$ is increasing on the interval (a, b) because $f'(x) > 0$
$f(x)$ is decreasing on the interval (a, b)	$f(x)$ is decreasing on the interval (a, b) because $f'(x) < 0$

<u>Justifications of Relative Minimums/Maximums and Points of Inflection</u>

Sign charts are very commonly used in calculus classes and are a valuable tool for students to use when testing for relative extrema and points of inflection. However, a sign chart will never earn students any points on the AP exam. Students should use sign charts when appropriate to help make determinations, but they cannot be used as a justification or explanation on the exam.

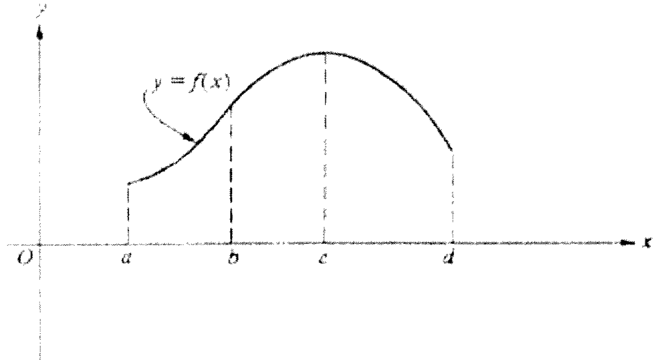
Situation (at a point $x = a$ on the function $f(x)$)	Proper Explanation/Reasoning
Relative Minimum	$f(x)$ has a relative minimum at the point $x = a$ because $f'(x)$ changes signs from negative to positive when $x = a$.
Relative Maximum	$f(x)$ has a relative maximum at the point $x = a$ because $f'(x)$ changes signs from positive to negative when $x = a$.
Point of Inflection	$f(x)$ has a point of inflection at the point $x = a$ because $f''(x)$ changes sign when $x = a$

Students need to be able to:

- Determine the graph of the function given the graph of its derivative and vice versa.
- Determine whether a function is increasing or decreasing using information about the derivative.
- Determine the concavity of a function's graph using information about the first or second derivative.
- Locate a function's relative and absolute extrema from its derivative.
- Locate a function's point(s) of inflection from its first or second derivative.
- Reason from a graph without finding an explicit rule that represents the graph.
- Sketch any of the related functions.
- Write justifications and explanations.
  - Must be written in sentence form.
  - Avoid using the pronoun "it" when justifying extrema.
  - Use "the function," "the derivative," or "the second derivative" instead of "the graph" or "the slope" in explanations.

Multiple Choice

1. (calculator not allowed)

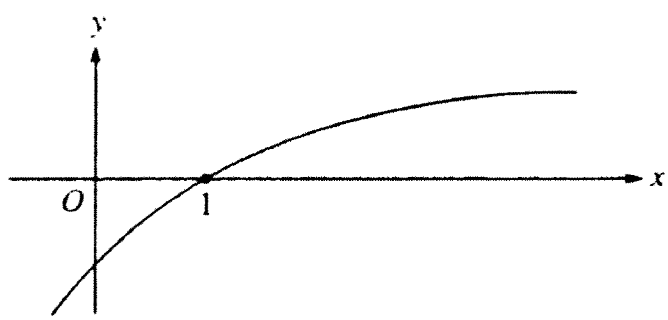


The graph of $y = f(x)$ is shown in the figure above. On which of the following intervals are $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$?

I. $a < x < b$
II. $b < x < c$
III. $c < x < d$

(A) I only (B) II only (C) III only (D) I and II (E) II and III

2. (calculator not allowed)



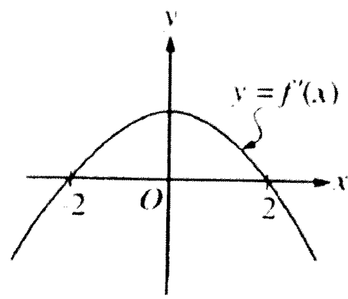
The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

(A) $f(1) < f'(1) < f''(1)$
(B) $f(1) < f''(1) < f'(1)$
(C) $f'(1) < f(1) < f''(1)$
(D) $f''(1) < f(1) < f'(1)$
(E) $f''(1) < f'(1) < f(1)$

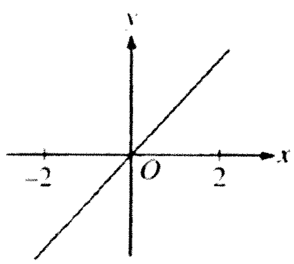
3. (calculator not allowed)
   If a function f is continuous for all x and if f has a relative maximum at $(-1, 4)$ and a relative minimum at $(3, -2)$, which of the following statements must be true?

   (A) The graph of f has a point of inflection somewhere between $x = -1$ and $x = 3$.
   (B) $f'(-1) = 0$
   (C) The graph of f has a horizontal asymptote.
   (D) The graph of f has a horizontal tangent line at $x = 3$.
   (E) The graph of f intersects both axes.

4. (calculator not allowed)



The graph of the derivative of f is shown in the figure above. Which of the following could be the graph of f?

(A)


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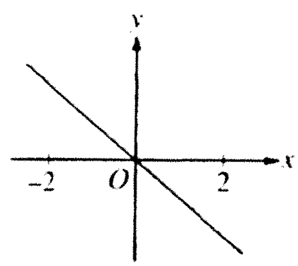

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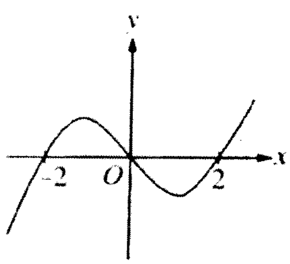

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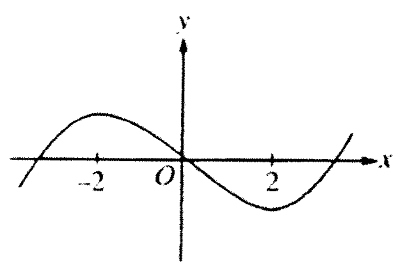

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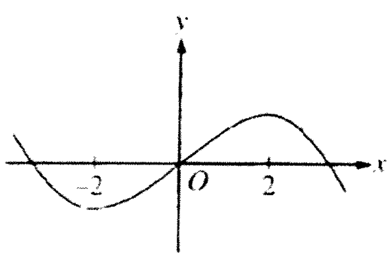


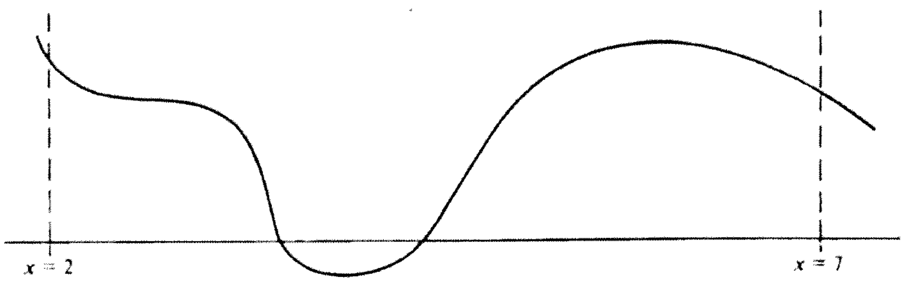
5. (calculator not allowed)

x	-4	-3	-2	-1	0	1	2	3	4
$g'(x)$	2	3	0	-3	-2	-1	0	3	2

The derivative g' of a function g is continuous and has exactly two zeros. Selected values of g' are given in the table above. If the domain of g is the set of all real numbers, then g is decreasing on which of the following intervals?

(A) $-2 \le x \le 2$ only
(B) $-1 \le x \le 1$ only
(C) $x \ge -2$ only
(D) $x \ge 2$ only
(E) $x \ge -2$ or $x \ge 2$

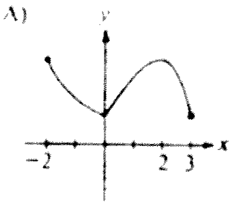
6. (calculator not allowed)



The graph of $y = f(x)$ on the closed interval $[2, 7]$ is shown above. How many points of inflection does this graph have on the interval

(A) One
(B) Two
(C) Three
(D) Four
(E) Five

7. (calculator not allowed)
Let f be a function that is continuous on the closed interval $[-2, 3]$ such that $f'(0)$ does not exist, $f'(2) = 0$, and $f''(x) < 0$ for all x except $x = 0$. Which of the following could be the graph of f?

(A)


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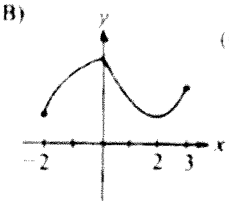

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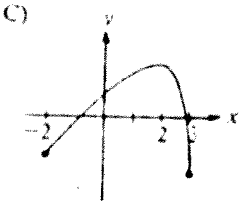

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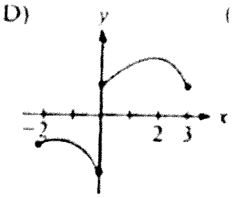

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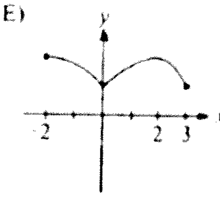
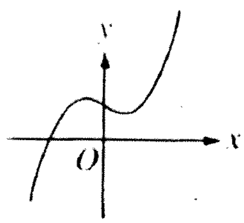
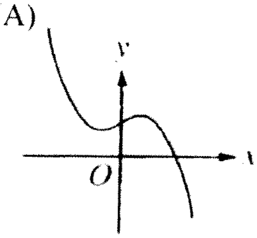


8. (calculator not allowed)



The graph $y = h(x)$ is shown above. Which of the following could be the graph of $y = h'(x)$?

(A)


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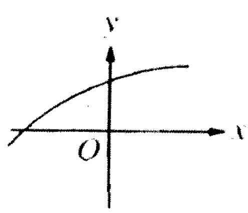
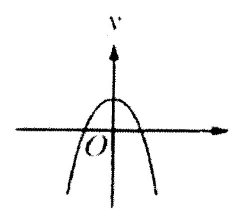


(C)


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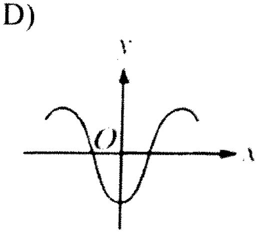

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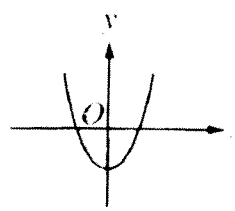
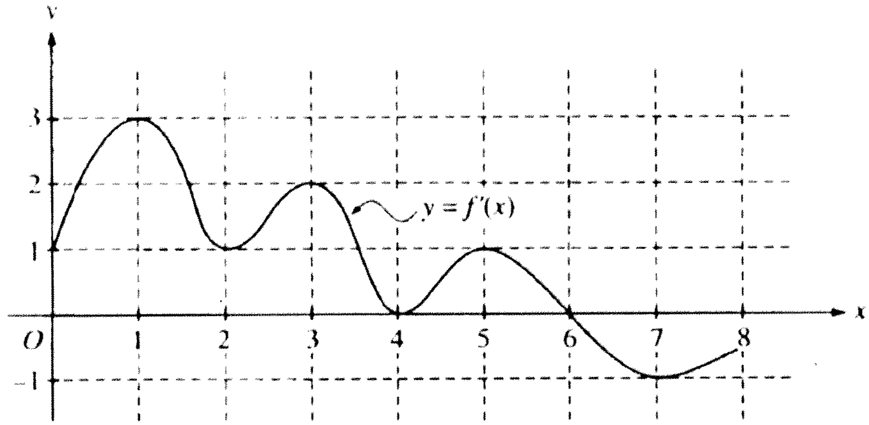


9. (calculator not allowed) (Use graph for 9-10)



The graph of f', the derivative of f, is given above. How many points of inflection does the graph of f have?

(A) Two
(B) Three
(C) Four
(D) Five
(E) Six

10. The point $(3, 5)$ is on the graph of $y = f(x)$. An equation of the line tangent to the graph of f at $(3, 5)$ is

(A) $y = 2$
(B) $y = 5$
(C) $y - 5 = 2(x - 3)$
(D) $y + 5 = 2(x - 3)$
(E) $y + 5 = 2(x + 3)$

11. (calculator not allowed)
Let g be a twice-differentiable function with $g'(x) > 0$ and $g''(x) > 0$ for all real numbers x, such that $g(4) = 12$ and $g(5) = 18$. Of the following, which is a possible value for $g(6)$?

(A) 15
(B) 18
(C) 21
(D) 24
(E) 27

12. (calculator allowed)
For all x in the closed interval $[2, 5]$, the function f has a positive first derivative and a negative second derivative. Which of the following could be a table for f?

(A)

x	2	3	4	5
$f(x)$	7	9	12	16

(B)

x	2	3	4	5
$f(x)$	7	11	14	16

(C)

x	2	3	4	5
$f(x)$	16	12	9	7

(D)

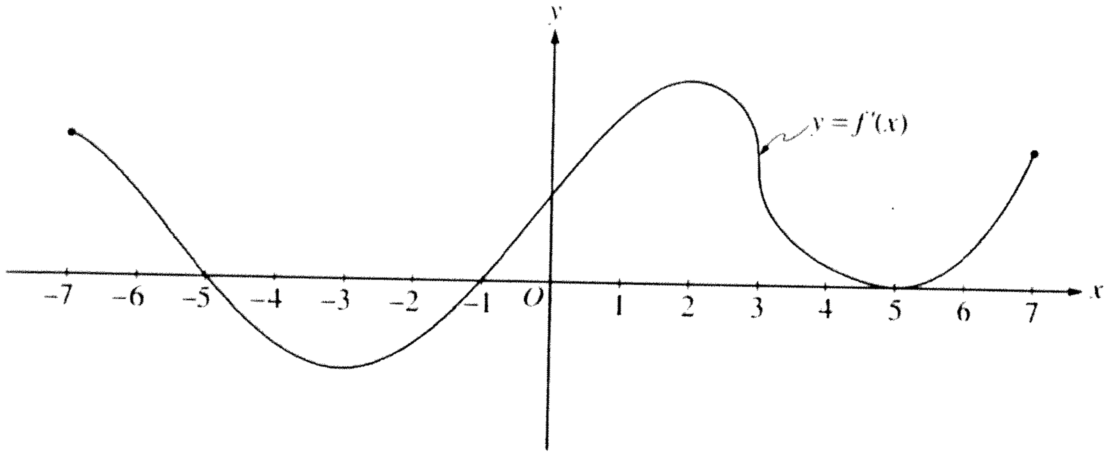
x	2	3	4	5
$f(x)$	16	14	11	7

(E)

x	2	3	4	5
$f(x)$	16	13	10	7

Free Response

13. (calculator allowed)



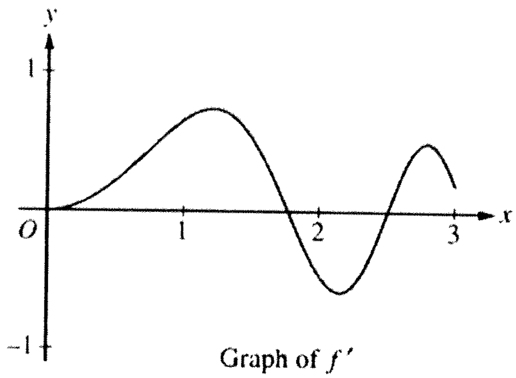
The figure above shows the graph of f', the derivative of f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

(a) Find all values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

(b) Find all values of x, for $-7 < x < 7$, at which f attains a relative maximum. Justify your answer.

(c) Find all values of x, for $-7 < x < 7$, at which $f'' < 0$.

14. (calculator allowed)



Graph of f'

Let f be the function defined for $x \geq 0$ with $f(0) = 5$ and f', the first derivative of f, given by $f'(x) = e^{(-x/4)} \sin\left(x^2\right)$. The graph of $y = f'(x)$ is shown above.

(a) Use the graph of f' to determine whether the graph of f is concave up, concave down, or neither on the interval $1.7 < x < 1.9$. Explain your reasoning.

15. (calculator not allowed)

Let f be a function that is <u>even</u> and continuous on the closed interval $[-3, 3]$. The function f and its derivatives have the properties indicated in the table below.

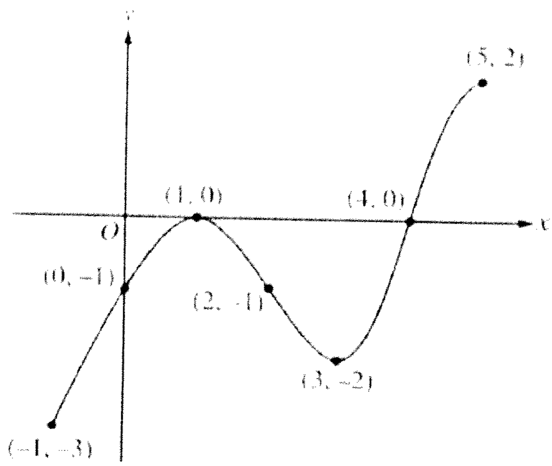
x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	1	Positive	0	Negative	-1	Negative
$f'(x)$	Undefined	Negative	0	Negative	Undefined	Positive
$f''(x)$	Undefined	Positive	0	Negative	Undefined	Negative

(a) Find the x-coordinate of each point at which f attains an absolute maximum value or an absolute minimum value. For each x-coordinate you give, state whether f attains an absolute maximum or an absolute minimum.

(b) Find the x-coordinate of each point of inflection on the graph of f. Justify your answer.

(c) In the xy-plane sketch the graph of a function with all the given characteristics of f.

16. (calculator not allowed)



Graph of f'

The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \le x \le 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.

(a) Find the x-coordinate of each of the points of inflection of the graph of f. Give a reason for your answer.