Analyzing f, f', and f''

Free response questions on this topic often require students to justify their answers. Students may use number lines to analyze the characteristics of the function; however, the justification must be written in sentence form. This justification should avoid using a pronoun such as "it" to describe the function and should make use of calculus involving the first or second derivative test or theorems.

Increase/Decrease

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on the interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on the interval.

Relative or Local Extrema – highest or lowest point in the neighborhood

- First derivative test
  - Candidates – critical numbers (x-values that make f' zero or undefined where f is defined)
  - Test – (1) set up an f' number line; label with candidates
    (2) test each section to see if f' is positive or negative
    (3) relative maximum occurs when f' changes from + to –
    relative minimum occurs when f' changes from – to +
- Second derivative test
  - Candidates – critical numbers (x-values that make f' zero or undefined where f is defined)
  - Test – (1) substitute each critical number into the second derivative
    (2) $f'' > 0$, relative minimum
    $f'' < 0$, relative maximum
    (3) $f'' = 0$, the test fails

Absolute or Global Extrema – highest or lowest point in the domain

- Absolute Extrema Test
  - Candidates – critical numbers and endpoints of the domain
  - Test – (1) find the y-values for each candidate
    (2) the absolute maximum value is the largest y-value,
    the absolute minimum value is the smallest y-value

Concavity

- If $f''(x) > 0$ (or $f'(x)$ is increasing) on an interval, then $f(x)$ is concave up on that interval.
- If $f''(x) < 0$ (or $f'(x)$ is decreasing) on an interval, then $f(x)$ is concave down on that interval.

Point of inflection – point where the concavity changes

- Determining points of inflection using the first derivative
  - The graph of f has a point of inflection where f' has a relative maximum or minimum
- Determining points of inflection using the second derivative
  - Candidates – x-values for which f'' is zero or undefined where f is defined
  - Test – (1) set up an f'' number line; label with candidates
    (2) test each section to see if f'' is positive or negative
    (3) any change in the sign of f'' indicates a point of inflection

Justifications for Increasing/Decreasing Intervals of a function

Remember: $f'(x)$ determines whether a function is increasing or decreasing, so always use the sign of $f'(x)$ when determining and justifying whether a function $f(x)$ is increasing or decreasing on (a, b).

Situation	Explanation
$f(x)$ is increasing on the interval (a, b)	$f(x)$ is increasing on the interval (a, b) because $f'(x) > 0$
$f(x)$ is decreasing on the interval (a, b)	$f(x)$ is decreasing on the interval (a, b) because $f'(x) < 0$

Justifications of Relative Minimums/Maximums and Points of Inflection

Sign charts are very commonly used in calculus classes and are a valuable tool for students to use when testing for relative extrema and points of inflection. However, a sign chart will never earn students any points on the AP exam. Students should use sign charts when appropriate to help make determinations, but they cannot be used as a justification or explanation on the exam.

Situation (at a point $x = a$ on the function $f(x)$)	Proper Explanation/Reasoning
Relative Minimum	$f(x)$ has a relative minimum at the point $x = a$ because $f'(x)$ changes signs from negative to positive when $x = a$.
Relative Maximum	$f(x)$ has a relative maximum at the point $x = a$ because $f'(x)$ changes signs from positive to negative when $x = a$.
Point of Inflection	$f(x)$ has a point of inflection at the point $x = a$ because $f''(x)$ changes sign when $x = a$

Students need to be able to:

- Locate critical numbers of the function and its derivatives.
- Determine whether a function is increasing or decreasing using information about the derivative.
- Determine the concavity of a function's graph using information about the first or second derivative.
- Locate a function's relative and absolute extrema from its derivative.
- Locate a function's point(s) of inflection from its first or second derivative.
- Sketch any of the related functions.
- Write justifications and explanations.
  - Must be written in sentence form.
  - Avoid using the pronoun "it" when justifying extrema.
  - Use "the function," "the derivative," or "the second derivative" instead of "the graph" or "the slope" in explanations.

Multiple Choice

1. (calculator not allowed)

   If $f(x)=1+x^{\frac{2}{3}}$, which of the following is NOT true?

   (A) f is continuous for all real numbers.
   (B) f has a minimum at $x=0$.
   (C) f is increasing for $x>0$.
   (D) $f'(x)$ exists for all x.
   (E) $f''(x)$ is negative for $x>0$.

2. (calculator not allowed)
   Given the function defined by $f(x)=3x^5-20x^3$, find all values of x for which the graph of f is concave up.

   (A) $x>0$
   (B) $-\sqrt{2}<x<0$ or $x>\sqrt{2}$
   (C) $-2<x<0$ or $x>2$
   (D) $x>\sqrt{2}$
   (E) $-2<x<2$

3. (calculator not allowed)
   What are all values of x for which the function f defined by $f(x)=\left(x^2-3\right)e^{-x}$ is increasing?

   (A) There are no such values of x.
   (B) $x<-1$ and $x>3$
   (C) $-3<x<1$
   (D) $-1<x<3$
   (E) All values of x

4. (calculator not allowed)
   If $f''(x) = x(x+1)(x-2)^2$, then the graph of f has inflection points when $x =$

   (A) -1 only
   (B) 2 only
   (C) -1 and 0 only
   (D) -1 and 2 only
   (E) -1, 0, and 2 only

5. (calculator not allowed)
   The function f is given by $f(x) = x^4 + x^2 - 2$. On which of the following intervals is f increasing?

   (A) $\left(-\frac{1}{\sqrt{2}}, \infty\right)$
   (B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
   (C) $(0, \infty)$
   (D) $(-\infty, 0)$
   (E) $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$

6. (calculator not allowed)
   The graph of $y = \dfrac{-5}{x-2}$ is concave downward for all values of x such that

   (A) $x < 0$
   (B) $x < 2$
   (C) $x < 5$
   (D) $x > 0$
   (E) $x > 2$

7. (calculator not allowed)
If $f(x) = x^2e^x$, then the graph of f is decreasing for all x such that

(A) $x < -2$ (B) $-2 < x < 0$ (C) $x > -2$ (D) $x < 0$ (E) $x > 0$

8. (calculator allowed)
Let f be the function given by $f(x) = \cos(2x) + \ln(3x)$. What is the least value of x at which the graph of f changes concavity?

(A) 0.56 (B) 0.93 (C) 1.18 (D) 2.38 (E) 2.44

9. (calculator allowed)

x	0	1	2	3	4
$f(x)$	2	3	4	3	2

The function f is continuous and differentiable on the closed interval $[0, 4]$. The table above gives selected values of f on this interval. Which of the following statements must be true?

(A) The minimum value of f on $[0, 4]$ is 2.

(B) The maximum value of f on $[0, 4]$ is 4.

(C) $f(x) > 0$ for $0 < x < 4$

(D) $f'(x) < 0$ for $2 < x < 4$

(E) There exists c, with $0 < c < 4$, for which $f'(c) = 0$.

10. (calculator allowed)

Let f be the function with derivative defined by $f'(x) = \sin(x^3)$ on the interval $-1.8 < x < 1.8$. How many points of inflection does the graph of f have on this interval?

(A) Two (B) Three (C) Four (D) Five (E) Six

11. (calculator allowed)

If the derivative of f is given by $f'(x) = e^x - 3x^2$, at which of the following values of x does f have a relative maximum value?

(A) -0.46
(B) 0.20
(C) 0.91
(D) 0.95
(E) 3.73

12. (calculator allowed)

If g is a differentiable function such that $g(x) < 0$ for all real numbers x and if $f'(x) = (x^2 - 4)g(x)$, which of the following is true?

(A) f has a relative maximum at $x = -2$ and a relative minimum at $x = 2$.
(B) f has a relative minimum at $x = -2$ and a relative maximum at $x = 2$.
(C) f has relative minima at $x = -2$ and at $x = 2$.
(D) f has relative maxima at $x = -2$ and at $x = 2$.
(E) It cannot be determined if f has any relative extrema.

Free Response

13. (calculator not allowed)

Given the function defined by $y = x + \sin x$ for all x such that $-\dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}$.

(a) Find the coordinates of <u>all</u> maximum and minimum points on the given interval. Justify your answers.

(b) Find the coordinates of <u>all</u> points of inflection on the given interval. Justify your answers.

(c) Sketch the graph of the function.

14. (calculator not allowed)

A function f is defined by $f(x) = xe^{2x}$ with domain $0 \le x \le 10$.

(a) Find all values of x for which the graph of f is increasing and all values of x for which the graph is decreasing.

(b) Give the x- and y-coordinates of all absolute maximum and minimum points on the graph of f. Justify your answers.

15. (calculator not allowed)

Suppose that the function f has a continuous second derivative for all x, and that $f(0) = 2$, $f'(0) = -3$, and $f''(0) = 0$. Let g be a function whose derivative is given by $g'(x) = e^{-2x}\left(3f(x) + 2f'(x)\right)$ for all x.

(a) Write an equation of the line tangent to the graph of f at the point where $x = 0$.

(b) Is there sufficient information to determine whether or not the graph of f has a point of inflection when $x = 0$? Explain your answer.

(c) Given that $g(0) = 4$, write an equation of the line tangent to the graph of g at the point where $x = 0$.

(d) Show that $g''(x) = e^{-2x}\left(-6f(x) - f'(x) + 2f''(x)\right)$. Does g have a local maximum at $x = 0$? Justify your answer.

16. (calculator not allowed)
Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of h at $x = 4$.

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?