Related Rates Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

Multiple Choice

1. E (1998 BC20 appropriate for AB)

   Let $f(x) = y = \sqrt[3]{x}$

   $$\frac{dy}{dt} = \frac{1}{3}x^{\frac{-2}{3}}\frac{dx}{dt}$$

   $$\frac{1}{3}x^{\frac{-2}{3}}\frac{dx}{dt} = \frac{1}{k}\frac{dx}{dt}$$

   When $x = 8$

   $$\frac{1}{3}(8)^{\frac{-2}{3}}\frac{dx}{dt} = \frac{1}{k}\frac{dx}{dt}$$

   $$\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{k}$$

   $$k = 12$$

2. E (1973 AB26)

   $$100\pi = 4\pi r^2; 2s = r^2; 5 = r$$

   $$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}; \frac{dV}{dt} = 4\pi(25)(.3) = 30\pi$$

3. C (1985 AB31)

   $$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt} + \frac{2}{3}\pi rh\frac{dr}{dt}$$

   $$\frac{dV}{dt} = \frac{1}{3}\pi(6^2)\left(\frac{1}{2}\right) + \frac{2}{3}\pi(54)\left(\frac{1}{2}\right) = 6\pi + 18\pi = 24\pi$$

4. B (1988 AB40)
Since

$$\frac{dx}{dt} = 3\frac{dy}{dt}, \frac{1}{3}\frac{dx}{dt} = \frac{dy}{dt}$$

$$x^2 + y^2 = z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$8\frac{dx}{dt} + 6\left(\frac{1}{3}\right)\frac{dx}{dt} = 10(1)$$

$$10\frac{dx}{dt} = 10$$

$$\frac{dx}{dt} = 1$$

5. D (1988 BC37 appropriate for AB)
Using similar triangles, $\frac{2}{x} = \frac{8}{z}$, $x =$ person's shadow, $z =$ streetlight shadow,

$$2z = 8x$$

$$2\frac{dz}{dt} = 8\frac{dx}{dt}$$

$$\frac{dz}{dt} = 4\frac{dx}{dt}$$

$$\frac{dz}{dt} = (4)\frac{4}{9} = \frac{16}{9}$$

$$\frac{d(z-x)}{dt} = \frac{dz}{dt} - \frac{dx}{dt} = \frac{16}{9} - \frac{4}{9} = \frac{4}{3}$$

6. A (1997 AB6)
Let

$$f(x) = y = \sqrt{x}$$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt}$$

$$\frac{1}{2\sqrt{c}}\frac{dx}{dt} = 2\left(\frac{1}{2}\right)\frac{dx}{dt}$$

$$\frac{1}{2\sqrt{c}} = 1$$

$$2\sqrt{c} = 1$$

$$\sqrt{c} = \frac{1}{2}; c = \frac{1}{4}$$

7. D (1998 AB90)

$$A = \frac{1}{2}bh$$

$$\frac{dA}{dt} = \frac{1}{2}b\frac{dh}{dt} + \frac{1}{2}h\frac{db}{dt}$$

$$\frac{dA}{dt} = \frac{-3b}{2} + \frac{3h}{2} < 0$$

if $b > h$

8. B (1998 AB78/BC78)

$$A = \pi r^2;\ \frac{dr}{dt} = -0.1\frac{\text{cm}}{\text{sec}}$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \text{ and } C = 2\pi r$$

$$\frac{dA}{dt} = C(-0.1) = -(0.1)C$$

9. A (1997 AB81)

$$x^2 + 70^2 = z^2$$

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt}$$

$$480\frac{dx}{dt} = 500\frac{dz}{dt}$$

$$\frac{480(60)}{500} = \frac{dz}{dt};\frac{dz}{dt} = \frac{288}{5} = 57.60$$

10. C (2003 AB78/BC78)

$$A = \pi r^2$$

$$20\pi = 2\pi r$$

$$r = 10$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(10)(.2) = 4\pi$$

Free Response

11. 2008 AB3ab

(a) When $r = 100\text{ cm}$ and $h = 0.5\text{ cm}$, $\frac{dV}{dt} = 2000 \text{ cm}^3/\text{min}$ and $\frac{dr}{dt} = 2.5 \text{ cm/min}$. $\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$ $\frac{dh}{dt} = 0.038$ or 0.039 cm/min	4: { 1: $\frac{dV}{dt} = 2000$ and $\frac{dr}{dt} = 2.5$ 2: expression for $\frac{dV}{dt}$ 1: answer }
(b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$. This occurs when $t = 25$ minutes. Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$, the oil slick reaches its maximum volume 25 minutes after the device begins working.	3: { 1: $R(t) = 2000$ 1: answer 1: justification }

12. 2007 AB5ab/BC5ab

(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$ $\left.\frac{dy}{dx}\right\|_{t=5} = 4\pi(30)^2(2) = 7200\pi \ \frac{\text{ft}^3}{\text{min}}$	3 { 2: $\frac{dV}{dt}$ 1: answer

13. 2007B AB3/BC3

(c) $\left.\frac{dW}{dt}\right\|_{t=3} = \left.\left(\frac{dW}{dv}\cdot\frac{dv}{dt}\right)\right\|_{t=3}$ $= W'(35)\cdot 5 = -0.892\,°\text{F/hr}$ or $W = 55.6 - 22.1(20+5t)^{0.16}$ $\left.\frac{dW}{dt}\right\|_{t=3} = -0.892\,°\text{F/hr}$	3 { 1: $\frac{dv}{dt} = 5$ 1: uses $v(3) = 35$ or uses $v(t) = 20 + 5t$ 1: answer
Units of °F/mph in (a) and °F/hr in (c)	1: units in (a) and (c)

14. 1995 AB5/BC3

(a) $\frac{r}{h}=\frac{4}{12}=\frac{1}{3}$ $r=\frac{1}{3}h$ $V=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h=\frac{\pi h^3}{27}$	2: 1: $r=\frac{1}{3}h$ 1: V as a function of h Note: 0/2 if r constant
(b) $\frac{dV}{dt}=\frac{\pi h^2}{9}\frac{dh}{dt}=\frac{\pi h^2}{9}(h-12)=-9\pi$ V is decreasing at $9\pi\ \text{ft}^3/\text{min}$	3: 1: $\frac{dV}{dt}$ using chain rule 1: $\frac{dh}{dt}=h-12$ 1: Solves for $\frac{dV}{dt}$ and gives answers with units Note: 0/1 if $\frac{dV}{dt}>0$
(c) Let $W=$ volume of water in the cylindrical tank $W=400\pi y$ $\frac{dW}{dt}=400\pi\frac{dy}{dt}$ $400\pi\frac{dy}{dt}=9\pi$ y is increasing at $\frac{9}{400}$ ft/min	4: 1: W is a function of y 1: $\frac{dW}{dt}=400\pi\frac{dy}{dt}$ 1: $\frac{dW}{dt}=\left\|\frac{dV}{dt}\right\|$ or $-\left\|\frac{dV}{dt}\right\|$ 1: Solves for $\frac{dy}{dt}$ and gives answer with unites

15. 2002B AB6

(a) Distance $= \sqrt{3^2 + 4^2} = 5$ km	1: answer
(b) $r^2 = x^2 + y^2$ $2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ or explicitly: $r = \sqrt{x^2 + y^2}$ $\frac{dr}{dt} = \frac{1}{2\sqrt{x^2 + y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$ At $x = 4$ and $y = 3$, $\frac{dr}{dt} = \frac{4(-15) + 3(10)}{5} = -6$ km/hr	4: { 1: expression for distance 2: differentiation with respect to t $< -2 >$ chain rule error 1: evaluation
(c) $\tan\theta = \frac{y}{x}$ $\sec^2\theta\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$ At $x = 4$ and $y = 3$, $\sec\theta = \frac{5}{4}$ $\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4) - (-15)(3)}{16}\right)$ $= \frac{85}{25} = \frac{17}{5}$ radians/hr	4: { 1: expression for θ in terms of x and y 2: differentiation with respect to t $< -2 >$ chain rule, quotient rule, or transcendental function error note: 0/2 if no trig or inverse trig function 1: evaluation