Optimization Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

Multiple Choice Questions:

1. B (1985 AB16)

$f'(x) = 3x^2 - 6x$

$3x(x-2) = 0$

$x = 0$ and $x = 2$

$f'(x) > 0$ for $(-\infty, 0)$, $f'(x) < 0$ for $(0, 2)$, $f'(x) > 0$ for $(2, \infty)$

f' changes from positive to negative only at $x = 0$.

Alternative:

$f''(x) = 6x - 6$

$f''(0) < 0$ so f is concave down at $x = 0$ and therefore a relative maximum.

$f''(2) > 0$ so f is concave up at $x = 2$ and therefore a relative minimum.

2. E (1973 BC27 appropriate for AB)

$f'(x) = x^2 - 8x + 12$

$(x-2)(x-6) = 0$

$x = 2,\ 6$

x	0	2	6	9
$f(x)$	–5	$\frac{17}{3}$	–5	22

The maximum value occurs at $x = 9$

3. D (1988 AB45)

$V = \pi r^2 h$

$16\pi = \pi r^2 h$

$h = \dfrac{16}{r^2}$

$A = 2\pi r h + 2\pi r^2$

$A = 2\pi r\left(\dfrac{16}{r^2}\right) + 2\pi r^2 = 2\pi r^2 + \dfrac{32\pi}{r} = 2\pi\left(r^2 + 16r^{-1}\right)$

$\dfrac{dA}{dr} = 2\pi(2r - 16r^{-2}) = 4\pi r^{-2}(r^3 - 8)$

$4\pi r^{-2}(r^3 - 8) = 0$

$r = 0,\ 2$

$\dfrac{dA}{dr} < 0$ for $0 < r < 2$ and $\dfrac{dA}{dr} > 0$ for $r > 2$, The minimum surface area of the can is when $r = 2$ and $h = 4$

4. D (1993 AB15)

$f'(x) = 2(x-2)(x-3) + (x-3)^2$

$f'(x) = (x-3)(2(x-2) + (x-3)) = (x-3)(3x-7)$

$0 = (x-3)(3x-7)$

$x = \frac{7}{3}, 3$

$f'(x) > 0$ for $\left(-\infty, \frac{7}{3}\right)$, $f'(x) < 0$ for $\left(\frac{7}{3}, 3\right)$, $f'(x) > 0$ for $(3, \infty)$

f' changes from positive to negative only at $x = \frac{7}{3}$.

5. C (1993 AB44)

$f'(x) = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$

$1 + \ln x = 0$

$\ln x = -1$

$x = e^{-1}$

f' changes from negative to positive only at $x = e^{-1}$, so $f(e^{-1}) = -\frac{1}{e}$.

Alternative:

$f''(x) = \frac{1}{x}$

$f''(e^{-1}) > 0$ so f is concave up at $x = e^{-1}$ and therefore a relative minimum.

6. B (1993 BC14 appropriate for AB)

$f'(x) = 0$

$x = 0, 2, -3$

$f'(x) > 0$ for $(-\infty, -3)$ and $(2, \infty)$

$f'(x) < 0$ for $(-3, 0)$ and $(0, 2)$

f' changes from positive to negative only at $x = -3$ creating only one relative maximum.

7. E (1993 BC36 appropriate for AB)

$h + 2\pi r = 30$

$h = 30 - 2\pi r$

$V = \pi r^2 h$

$V = \pi r^2(30 - 2\pi r) = 2\pi(15r^2 - \pi r^3)$ for $0 < r < \frac{15}{\pi}$

$\frac{dV}{dr} = 2\pi(30r - 3\pi r^2)$

$r = \frac{10}{\pi}$

$\frac{dV}{dr} > 0$ for $\left(0, \frac{10}{\pi}\right)$ and $\frac{dV}{dr} < 0$ for $\left(\frac{10}{\pi}, \frac{15}{\pi}\right)$, creating a maximum volume at $r = \frac{10}{\pi}$.

8. B (1969 AB11/BC11)

Let L be the distance from $\left(x, -\frac{x^2}{2}\right)$ and $\left(0, -\frac{1}{2}\right)$ such that

$$L^2 = (x-0)^2 + \left(\frac{x^2}{2} - \frac{1}{2}\right)^2$$

$$2L\frac{dL}{dx} = 2x + 2\left(\frac{x^2}{2} - \frac{1}{2}\right)(x)$$

$$2L\frac{dL}{dx} = 2x + x^3 - x$$

$$2L\frac{dL}{dx} = x + x^3$$

$$\frac{dL}{dx} = \frac{x(1+x^2)}{2L} = 0$$

$$x = 0$$

$\frac{dL}{dx} < 0$ for all $x < 0$ and $\frac{dL}{dx} > 0$ for all $x > 0$, so the minimum distance occurs at $x = 0$

$$0^2 + 2y = 0 \Rightarrow y = 0$$

9. A (1997 BC9 appropriate for AB)

f increases for $0 \le x \le 6$ ($f' \ge 0$) and f decreases $6 \le x \le 8$ ($f' < 0$). By comparing the areas it is clear that f increases more than it decreases, so the absolute minimum must occur at the left endpoint, $x = 0$.

10. B (1988 BC45 appropriate for AB)

$A = (2x)(2y) = 4xy$ and $y = \sqrt{4 - \frac{4}{9}x^2}$, so $A = 8x\sqrt{1 - \frac{1}{9}x^2}$,

$$A' = 8\left(\left(1 - \frac{1}{9}x^2\right)^{\frac{1}{2}} + \frac{1}{2}x\left(1 - \frac{1}{9}x^2\right)^{-\frac{1}{2}}\left(-\frac{2}{9}x\right)\right) = \frac{8}{9}\left(1 - \frac{1}{9}x^2\right)^{-\frac{1}{2}}\left(9 - 2x^2\right)$$

$A' = 0$ at $\frac{3}{\sqrt{2}}$; A' is undefined at $x = \pm 3$. The maximum area occurs when $x = \frac{3}{\sqrt{2}}$ and $y = \sqrt{2}$. The value of the largest area is $A = 4xy = 4\left(\frac{3}{\sqrt{2}}\right)\left(\sqrt{2}\right) = 12$

11. B (1988 AB33)

Let $P(x) = x(2x-8) = 2x^2 - 8x$

$P'(x) = 4x - 8$

$0 = 4x - 8$

$x = 2$

$P''(x) = 4$ second derivative is positive for all x values, therefore $P(x)$ has a minimum at $x = 2$; $P(2) = 2(2)^2 - 8(2) = -8$

Alternatively:

P' changes from negative to positive at $x = 2$; $P(2) = 2(2)^2 - 8(2) = -8$

Free Response

12. (calculator not allowed) (2008 AB6b)

(b) $f'(x) = 0$ when $x = e$. The function f has a relative maximum at $x = e$ because $f'(x)$ changes from positive to negative at $x = e$.	3 { 1: $x = e$ 1: relative maximum 1: justification

13. (calculator allowed) (2009 AB2b)

(b) $R'(t) = 0$ when $t = 0$ and $t = 1.36296$ The maximum rate may occur at 0, $a = 1.36296$, or 2. $R(0) = 0$ $R(a) = 854.527$ $R(2) = 120$ The maximum rate occurs when $t = 1.362$ or 1.363.	3 { 1: consider $R'(t) = 0$ 1: interior critical point 1: answer and justification

14. (calculator allowed) (2010 AB2d)

(d) $P'(t) = 0$ when $t = 9.183503$ and $t = 10.816497$.

t	$P(t)$
8	0
9.183503	5.088662
10.816497	2.911338
12	8

Entries are being processed most quickly at time $t = 12$.

3 { 1: considers $P'(t) = 0$
1: identifies candidates
1: answer with justification

15. (calculator not allowed) (2010 AB5c)

(c) $h'(x) = g'(x) - x = 0 \Rightarrow g'(x) = x$ On the interval $-2 \le x \le 2$, $g'(x) = \sqrt{4 - x^2}$. On this interval, $g'(x) = x$ when $x = \sqrt{2}$. The only other solution to $g'(x) = x$ is $x = 3$. $h'(x) = g'(x) - x > 0$ for $0 \le x < \sqrt{2}$ $h'(x) = g'(x) - x \le 0$ for $\sqrt{2} < x \le 5$ Therefore h has a relative maximum at $x = \sqrt{2}$ and h has neither a minimum nor a maximum at $x = 3$.	4: { 1: $h'(x)$ 1: identifies $x = \sqrt{2}$, 3 1: answer for $\sqrt{2}$ with analysis 1: answer for 3 with analysis

16. (calculator not allowed) (2009 AB6c)

(c) Since $f'(x) > 0$ on the intervals $-4 < x < -2$ and $-2 < x < 3\ln\left(\frac{5}{3}\right)$, f is increasing on the interval $-4 \le x \le 3\ln\left(\frac{5}{3}\right)$. Since $f'(x) < 0$ on the interval $3\ln\left(\frac{5}{3}\right) < x < 4$, f is decreasing on the interval $3\ln\left(\frac{5}{3}\right) \le x \le 4$. Therefore, f has an absolute maximum at $x = 3\ln\left(\frac{5}{3}\right)$.	2: { 1: answer 1: justification

17 2007 AB2/BC2

(c) Since $f(t)-g(t)$ changes sign from positive to negative only at $t=3$, the candidates for the absolute maximum are at $t=0$, 3, and 7.

t (hours)	Gallons of water
0	5000
3	$5000+\int_0^3 f(t)\,dt-250(3)=5126.591$
7	$5126.591+\int_3^7 f(t)\,dt-2000(4)=4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

5:
- 1: identifies $t=3$ as a candidate
- 1: integrand
- 1: amount of water at $t=3$
- 1: amount of water at $t=7$
- 1: conclusion