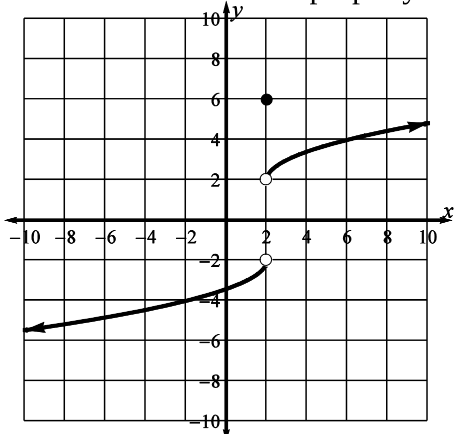
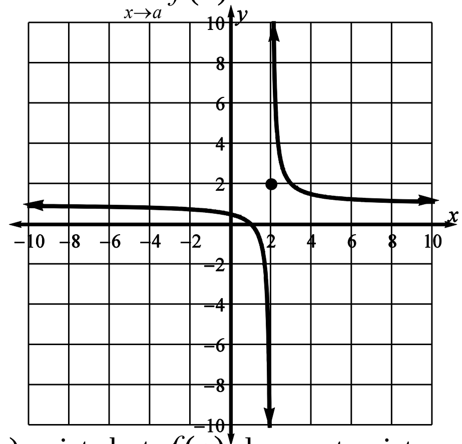
Limits, Continuity, and Differentiability Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

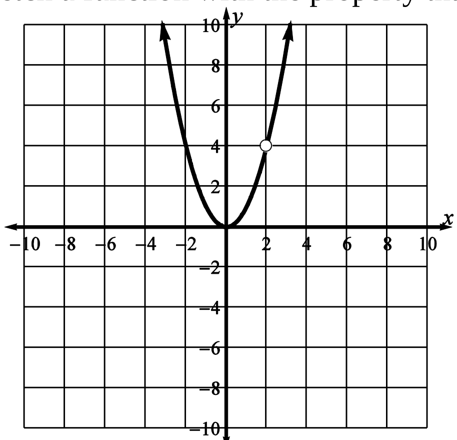
Quick Check for Understanding: *Student graphs will vary. Sample answers given.*

1. Sketch a function with the property that $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exist.

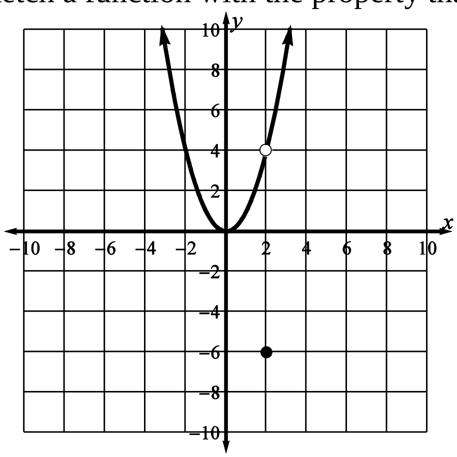




2. Sketch a function with the property that $\lim_{x \to a} f(x)$ exists but $f(a)$ does not exist.



3. Sketch a function with the property that $f(a)$ exists and $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$.



Multiple Choice

1. D (1985 AB5)
Since the limit is taken as $n \to \infty$ and the exponents in the numerator and denominator are equal, use the ratio of the leading coefficients to find that the limit is $\frac{4n^2}{n^2} = 4$.

2. B (2008 AB1)
Multiplying in the numerator and denominator yields the equivalent limit:
$\lim_{x\to\infty} \frac{-2x^2+7x-3}{x^2+2x-3} = -2$.

3. B (1969 AB6/BC6)
This limit represents the definition of the derivative of the function, $f(x) = 8x^8$ at $x = \frac{1}{2}$.
$f'(x)=64x^{7}$; $f'\left(\frac{1}{2}\right) = \frac{1}{2}$.

4. D (1985 BC29 appropriate for AB)
Let $x - \frac{\pi}{4} = t$. $\lim_{x\to\frac{\pi}{4}} \frac{\sin\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}} = \lim_{t\to 0} \frac{\sin t}{t} = 1$.

5. B (1988 AB29)
This limit represents the derivative of the function, $f(x) = \tan 3x$. Using the chain rule, $f'(x) = 3\sec^2(3x)$

6. C (1993 BC2 appropriate for AB)
$\lim_{x\to 0} \frac{2x^2+1-1}{x^2} = 2$

7. B (1988 AB23)
Since $f'(x) = \cos x$, $f(x) = \sin x$. Also $f(0) = 0$, $g(0) = 0$, and $g'(x) = 1$, hence
$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\sin x}{x} = 1$.
Alternatively, by L'Hopital's rule, $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{f'(0)}{g'(0)} = \frac{\cos 0}{1} = 1$.

8. C (1993 AB29)
$\lim_{x\to 0} \frac{1-\cos\theta}{2\sin^2\theta} = \lim_{x\to 0} \frac{1-\cos\theta}{2(1-\cos^2\theta)} = \lim_{x\to 0} \frac{1-\cos\theta}{2(1-\cos\theta)(1+\cos\theta)} = \lim_{x\to 0} \frac{1}{2(1+\cos\theta)} = \frac{1}{4}$.
Alternatively, by L'Hopital's rule, $\lim_{x\to 0} \frac{1-\cos\theta}{2\sin^2\theta} = \lim_{x\to 0} \frac{\sin\theta}{4\sin\theta\cos\theta} = \lim_{x\to 0} \frac{1}{4\cos\theta} = \frac{1}{4}$.

.

9. E (1985 AB41)
Using the given limit, there is not enough information to establish that $f'(a)$ exists, nor that $f(x)$ is continuous or defined at $x = a$, nor that $f(a) = L$. For example, consider the function whose graph is the horizontal line $y = 2$ with a hole at $x = a$. For this function $\lim_{x\to a} f(x) = 2$ and none of the given statements are true

10. E (2003 AB3)
By definition of a horizontal asymptote, E is correct.

11. E (1993 AB35)
Since $y = 2$ is a horizontal asymptote, the ratio of the leading coefficients must be 2; therefore, $a = 2$. Since there is a vertical asymptote at $x = -3$, set the denominator, $-3 + c$, equal to 0, so $c = 3$ then $a + c = 2 + 3 = 5$.

12. E (1988 AB27)
$f(3) = 6(3) - 9 = 9$

$\lim_{x\to 3^-} x^2 = \lim_{x\to 3^+} 6x - 9 = 9$

Since $f(3) = \lim_{x\to 3} f(x)$ the function is continuous at $x = 3$

$f'(x) = \begin{cases} 2x, & x < 3 \\ 6, & x > 3 \end{cases}$ and $\lim_{x\to 3^-} 2x = \lim_{x\to 3^+} 6 = 6$

Since the left and right limits of the derivative of the function are equivalent from either side of 3, the function is differentiable at $x = 3$.

13. D (2003 AB20)
Using substitution, the one sided limits as $x \to 3$ are both equal to 5; therefore, I and II are true. f is not differentiable at $x = 3$ since $f'(3) = 1$ for $x \le 3$ and $f'(3) = 4$ for $x > 3$.

14. A (2008 AB6/BC6)
Use the top piece of the piecewise function for the limit since the bottom piece gives the value of $f(2)$. Using factoring, $\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x + 2 = 4$. Since this limit does not equal $f(2) = 1$, the function f is not continuous or differentiable at $x = 2$.

15. C (1988 BC5 appropriate for AB)
$\lim_{x\to a} f(x) = f(a)$ for all values of a except 2. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x - 2) = 0 \ne 1 = f(2)$.

16. E (1969 AB18/BC18)
Thinking graphically, the absolute value function will have a sharp corner at 3, thus the derivative does not exist at that point. The question can also be worked algebraically as follows:
The one sided limits are not equal:
$\lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = \frac{2+|x-3|-2}{x-3} = -1$
$\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \frac{2+|x-3|-2}{x-3} = 1$.
Therefore, the value of $f'(3)$ does not exist.

17. A (1993 AB5)
Consider $f(a) = \lim_{x\to a} f(x)$
Use factoring to simplify the function and then substitute for x:
$\lim_{x\to 2} \frac{x^2-4}{x+2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x+2} = x+2$; therefore, $f(-2) = -4$.

18. B (1997 AB15)
The left and right limits as $x \to a$ are both equal to 2. The limit as $x \to b$ does not exist since the one-sided limits are not equivalent; therefore, A, C and D cannot be true either.

19. A (2003 AB13/BC13)
The graph of f is continuous at $x = a$; however, since the graph has a sharp turn at $x = a$, the function is not differentiable at $x = a$.

20. D (2003 AB79)
The one-sided limits as $x \to 4$ are equivalent for the graphs of f in I and II but not for III.

21. C (2008 AB77)
$\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^-} f(x)$ exist; however, since they are not equivalent, the $\lim_{x\to 2} f(x)$ does not exist.

22. A (2003 BC81 appropriate for AB)
$\sin(2) \approx 0.9093$

Free Response

23. 2009B AB3abc/BC3abc

(a) $\lim_{x\to 0^-} \frac{f(h)-f(0)}{h} = \frac{2}{3}$ $\lim_{x\to 0^+} \frac{f(h)-f(0)}{h} < 0$ Since the one-sided limits do not agree f is not differentiable at $x=0$.	2 { 1: sets up difference quotient at $x=0$ 1: answer with justification
(b) $\frac{f(6)-f(a)}{6-a} = 0$ when $f(a)=f(6)$. There are two values of *a* for which this is true.	2 { 1: expression for average rate of change 1: answer with reason
(c) Yes, $a=3$. The function f is differentiable on the interval $3<x<6$ and continous on $3\le x\le 6$. Also, $\frac{f(6)-f(3)}{6-3} = \frac{1-0}{6-3} = \frac{1}{3}$. By the Mean Value Theorem, there is a value *c*, $3<x<6$, such that $f'(c)=\frac{1}{3}$.	2 { 1: answers "yes" and identifies $a=3$ 1: justification

24. 2003 AB6ac

(a) *f* is continuous at $x=3$ because $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 2.$ Therefore, $\lim_{x\to 3} f(x) = 2 = f(3).$	2 { 1: answer "yes" and equates the values of the left- and right-hand limits 1: explanation involving limits
(c) Since *g* is continuous at $x=3$, $2k=3m+2$. $g'(x) = \begin{cases} \frac{k}{2\sqrt{x+1}}; & 0<x<3 \\ m\ ; & 3<x<5 \end{cases}$ $\lim_{x\to 3^-} g'(x) = \frac{k}{4}$ and $\lim_{x\to 3^+} g'(x) = m$ Since these two limits exist and *g* is differentiable at $x=3$, the two limits are equal. Thus $\frac{k}{4}=m.$ $8m=3m+2;\ m=\frac{2}{5}$ and $k=\frac{8}{5}$	3 { 1: $2k=3m+2$ 1: $\frac{k}{4}=m$ 1: values for *k* and *m*

25. 2011 AB6ab

(a) $\lim_{x\to 0^-}(1-2\sin x)=1$ $\lim_{x\to 0^+} e^{-4x}=1$ $f(0)=1$ So, $\lim_{x\to 0} f(x)=f(0)$. Therefore *f* is continuous at $x=0$.	2: analysis
(b) $f'(x)=\begin{cases}-2\cos x & \text{for } x<0\\ -4e^{-4x} & \text{for } x>0\end{cases}$ $-2\cos x \ne -3$ for all values of $x<0$. $-4e^{-4x}=-3$ when $x=-\frac{1}{4}\ln\left(\frac{3}{4}\right)>0$. Therefore $f'(x)=-3$ for $x=-\frac{1}{4}\ln\left(\frac{3}{4}\right)$.	3 { 2: $f'(x)$ 1: value of x

Student samples for question 25a

Sample A:
Student earned both points for part (a). The student clearly identified the three attributes required to be true to justify continuity.

Sample B:
Student earned 1 point for part (a). The student begins the analysis of continuity by looking at the functional values on each side of 1. The student does not use limits and does not consider $f(0)=1$, thus earning only 1 of the possible 2 points.