Fundamental Theorem of Calculus Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

Multiple Choice Questions Solutions

1. A (1985 AB22)

$$\int_1^2 \frac{(x-1)(x+1)}{x+1}\,dx = \int_1^2 (x-1)\,dx$$

$$\left(\frac{1}{2}x^2 - x\right)\Bigg|_{x=1}^{x=2} = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

2. C (1988 AB10)

$$\left(\frac{2kx^2}{2} - \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=k} = 18$$

$$\left(k^3 - \frac{k^3}{3}\right) = 18$$

$$\frac{2}{3}k^3 = 18$$

$$k = 3$$

3. C (1988 AB36)

The first quadrant interval for $y = 3x - x^2$ is $[0, 3]$.

$$y_{avg} = \frac{\int_0^3 \left(3x - x^2\right)dx}{3-0}$$

$$y_{avg} = \frac{\left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{x=0}^{x=3}}{3}$$

$$y_{avg} = \frac{\left(\frac{27}{2} - 9\right) - 0}{3} = \frac{3}{2}$$

4. A (1988 AB13)

Using the Fundamental Theorem of Calculus, $\int_a^b f'(x)dx = f(b) - f(a)$, it follows directly that $\int_0^c f'(x)\,dx = f(c) - f(0)$.

5. D (2003 AB22)

$f(1) = f(0) + \int_0^1 f'(x)\,dx = 5 + 3 = 8$

Alternatively, the equation for the derivative shown is $f'(x) = -6x + 6$.

$f(x) = \int(-6x+6)dx$

$f(x) = -3x^2 + 6x + c$

With $f(0) = 5$ implies $c = 5$ and therefore $f(1) = 3(1)^2 - 6(1) + 5 = 8$

6. E (2003 AB23)

Applying the Second Fundamental Theorem, $\dfrac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))\,g'(x)$

$\dfrac{d}{dx}\int_0^{x^2} \sin(t^3)dt = \left(\sin(x^2)^3\right)(2x) = 2x\sin(x^6)$

7. C (1993 BC41 appropriate for AB)

$f'(x) = \dfrac{d}{dx}\int_{-2}^{x^2-3x} e^{t^2}\,dt = e^{(x^2-3x)^2}(2x-3)$

$f'(x) = 0$ only when $x = \dfrac{3}{2}$.

$f' < 0$ for $\left(-\infty, \dfrac{3}{2}\right)$ and $f' > 0$ for $\left(\dfrac{3}{2}, \infty\right)$ thus $f(x)$ has a minimum at $x = \dfrac{3}{2}$.

8. D (1988 AB14)

$\int_0^{\frac{\pi}{2}} \dfrac{\cos\theta}{\sqrt{1+\sin\theta}}\,d\theta$

Let $u = 1 + \sin\theta$

$\int_1^2 u^{\frac{-1}{2}}\,du = -2u^{\frac{1}{2}}\Big|_{u=1}^{u=2} = -2\sqrt{2} - (-2)$

9. C (2003 BC18 appropriate for AB)

$g'(x) = \dfrac{d}{dx}\int_0^{2x} f(t)\,dt = 2f(2x)$

$g'(3) = 2f(2\cdot 3) = 2f(6) = 2(-1) = -2$

10. E (1993 BC3 appropriate for AB)

$Q'(x) = p(x)$, so the degree of Q is $n+1$.

11. E (1973 BC45 appropriate for AB)
$F'(x) = xg'(x)$ with $x \geq 0$ and $g'(x) < 0$, so $F'(x) < 0$; therefore, F is decreasing (not increasing).

12. E (2003 AB19)
$$\frac{d}{dx}\left(\int_0^{x^3} \ln\left(t^2+1\right) dt\right)$$
$$= (3x^2)\ln\left((x^3)^2+1\right)$$
$$= (3x^2)\ln\left(x^6+1\right)$$

13. B (1988 AB19)
$u = x^2+1$
$du = 2xdx$
$x = 2 \Rightarrow u = 2^2+1 = 5$ and $x = 3 \Rightarrow u = 3^2+1 = 10$, so
$$\frac{1}{2}\int_5^{10} \frac{1}{u} du$$
$$= \frac{1}{2}(\ln 10 - \ln 5)$$
$$= \frac{1}{2}\ln\frac{10}{5}$$
$$= \frac{1}{2}\ln 2$$

14. D (2003 AB92)
$g(x)$ is decreasing when $g'(x) < 0$.
$$g'(x) = \frac{d}{dx}\int_0^x \sin(t^2)dt$$
$g'(x) = \sin(x^2)$
Using a graphing calculator, determine where $g'(x) < 0$.
$1.772 \leq x \leq 2.507$

15. B (1997 BC82 appropriate for AB)
Since $\int_0^x (t^2-2t)\ dt \geq \int_2^x t\ dt$, $\frac{1}{3}x^3 - x^2 \geq \frac{1}{2}x^2 - 2$. Using the calculator, the greatest x-value on the interval $[0, 4]$ that satisfies this inequality is found to occur at $x = 1.3887$.

16. E (1997 AB88)
$f(x) = \int_a^x h(x)dx$, so $f(a) = 0$; therefore, only choices (A) and (E) are possible. But $f'(x) = h(x)$, so f is differentiable at $x = b$. This is true for the graph in option (E), but not for the graph in option (A), where there appears to be a sharp turn at $x = b$. Also, since h is increasing at first, the graph of f must start out concave up. This is also true in (E) but not (A).

17. A (2008 BC88 appropriate for AB)

$g(x)=\int_2^x f(t)dt$ and $f>0$ and $f'<0$; $g'(x)=f(x)$, so $g''(x)=f'(x)$. $g(2)=\int_2^2 f(t)\,dt=0$. Since $f>0$, $g'>0$, so g is increasing, so $f(1)<g(2)$, so $g(1)=-2$. $f'<0$, so $g''<0$, so g' is decreasing; therefore, $\frac{g(3)-g(2)}{3-2}<\frac{g(2)-g(1)}{2-1}$, so the answer is choice A.

18. A (2003 AB82)

The function $r(t)$ is the rate of change in the altitude, so the altitude is decreasing when $r(t)<0$. The zeros of $r(t)$ are 1.572 and 3.514, so the change in altitude when the altitude is decreasing can be found using $\int_{1.572}^{3.514} r(t)\,dt$.

19. A (2003 AB84)

$T(5)=T(0)+\int_0^5(-110e^{-0.4t})\;dt=350-237.78\approx 112^\circ F$

20. A (2003 BC80 appropriate for AB)

$T=\int_7^{14}\frac{100e^{-0.1t}}{2-e^{-3t}}\;dt=124.499\approx 125\text{ tons}$

Free Response Questions

21. 2010 AB1/BC1acd

(a) $\int_0^6 f(t)\,dt = 142.274$ or 142.275 cubic feet	$2\begin{cases}1: & \text{integral}\\1: & \text{answer}\end{cases}$
(c) $h(0)=0$ For $0<t\le 6$, $h(t)=h(0)+\int_0^t g(s)\,ds = 0+\int_0^t 0\,ds = 0$ For $6<t\le 7$, $h(t)=h(6)+\int_6^t g(s)\,ds = 0+\int_6^t 125\,ds = 125(t-6)$ For $7<t\le 9$, $h(t)=h(7)+\int_7^t g(s)\,ds = 125+\int_7^t 108\,ds = 125+108(t-7)$ Thus, $h(t)=\begin{cases}0; & 0\le t\le 6\\125(t-6); & 6<t\le 7\\125+108(t-7); & 7<t\le 9\end{cases}$	$3\begin{cases}1: & h(t) \text{ for } 0\le t\le 6\\1: & h(t) \text{ for } 6<t\le 7\\1: & h(t) \text{ for } 7<t\le 9\end{cases}$
(d) Amount of snow is $\int_0^9 f(t)\,dt - h(9) = 26.334$ or 26.335 cubic feet.	$3\begin{cases}1: & \text{integral}\\1: & h(9)\\1: & \text{answer}\end{cases}$

22. 2009 AB2/BC2acd

(a) $\int_0^2 R(t)\,dt = 980$ people	$2\begin{cases}1: & \text{integral}\\1: & \text{answer}\end{cases}$
(c) $w(2)-w(1)=\int_1^2 w'(t)\,dt = \int_1^2 (2-t)R(t)\,dt = 387.5$ The total wait time for those who enter the auditorium after time $t=1$ is 387.5 hours.	$2\begin{cases}1: & \text{integral}\\1: & \text{answer}\end{cases}$
(d) $\frac{1}{980}w(2)=\frac{1}{980}\int_0^2 (2-t)R(t)\,dt = 0.77551$ On average, a person waits 0.775 or 0.776 hour.	$2\begin{cases}1: & \text{integral}\\1: & \text{answer}\end{cases}$

23. 2009 AB5/BC5b

(b) $\int_2^{13}\left(3-5f'(x)\right)dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$ $= 3(13-2) - 5\left(f(13) - f(2)\right) = 8$	$2\begin{cases}1: \text{ uses Fundamental Theorem of Calculus}\\ 1: \text{ answer}\end{cases}$

24. 2010 AB5a

(a) $g(3) = 5 + \int_0^3 g'(x)\,dx = 5 + \frac{\pi \cdot 2^2}{4} + \frac{3}{2} = \frac{13}{2} + \pi$ $g(-2) = 5 + \int_0^{-2} g'(x)\,dx = 5 - \pi$	$3\begin{cases}1: \text{ uses } g(0)=5\\ 1: \ g(3)\\ 1: \ g(-2)\end{cases}$

25. 2008B AB4

(a) $f'(x) = 3\sqrt{4+(3x)^2}$ $g'(x) = f'(\sin x)\cdot\cos x$ $= \left(3\sqrt{4+(3\sin x)^2}\right)\cos x$	$4\begin{cases}2: \ f'(x)\\ 2: \ g'(x)\end{cases}$
(c) For $0 < x < \pi$, $g'(x) = 0$ only at $x = \frac{\pi}{2}$. $g(0) = g(\pi) = 0$ $g\left(\frac{\pi}{2}\right) = \int_0^3 \sqrt{4+t^2}\,dt > 0$ The maximum value of g on $[0, \pi]$ is $\int_0^3 \sqrt{4+t^2}\,dt$.	$3\begin{cases}1: \text{ sets } g'(x)=0\\ 1: \text{ justifies maximum at } \frac{\pi}{2}\\ 1: \text{ integral expression for } g\left(\frac{\pi}{2}\right)\end{cases}$

26. 2009B AB1ac

(a) $R(t)=6+\int_0^t \frac{1}{16}\left(3+\sin\left(x^2\right)\right)dx$ $R(3)=6.610$ or 6.611	3 { 1: integral 1: expression for $R(t)$ 1: $R(3)$
(c) $\int_0^3 A'(t)\,dt = A(3)-A(0)=24.200$ or 24.201 From time $t=0$ to $t=3$ years, the cross-sectional area grows by 24.201 square centimeters.	3 { 1: uses Fundamental Theorem of Calculus 1: value of $\int_0^3 A'(t)\,dt$ 1: meaning of $\int_0^3 A'(t)\,dt$

27. 2011B AB6

(a) $\int_{-2\pi}^{4\pi} f(x)\,dx = \int_{-2\pi}^{4\pi}\left(g(x)-\cos\left(\frac{x}{2}\right)\right)dx$ $=6\pi^2-\left[2\sin\left(\frac{x}{2}\right)\right]_{x=-2\pi}^{x=4\pi}$ $=6\pi^2$	2 { 1: antiderivative 1: answer
(c) $h'(x)=g(3x)\cdot 3$ $h'\left(-\frac{\pi}{3}\right)=3g(-\pi)=3\pi$	3 { 2: $h'(x)$ 1: answer