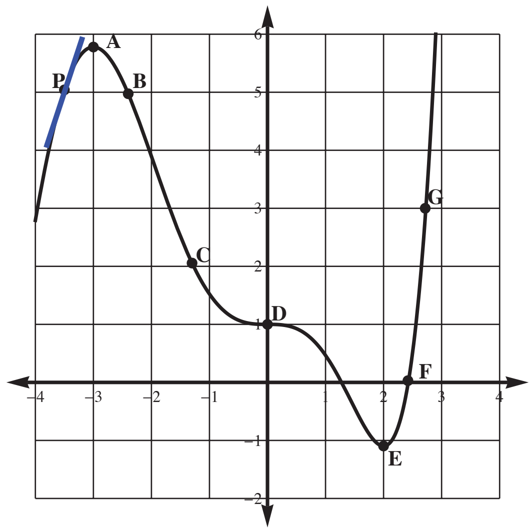

Average Rate of Change vs. Instantaneous Rate of Change

ABOUT THIS LESSON

This lesson introduces students to the concept of the slope of a non-linear function. Students use their prior knowledge of distance-time graphs and slopes of lines to investigate slopes of secant and tangent lines. Students draw a short line segment tangent to the function to approximate the function's instantaneous rate of change. The lesson begins with a scenario involving constant rate, and then students consider a case where the rate is not constant but the average rate remains the same. Through a series of definitions and questions, students draw conclusions about the implications of positive and negative rates of change, as well as about the relationship between position and velocity functions.

OBJECTIVES

Students will

- create distance-time graphs from given information.
- discover the difference between the average rate of change and the instantaneous rate of change of a function.
- approximate the instantaneous rate of change using short tangent line segments.
- interpret the sign and magnitude of the instantaneous rate of change in the context of the situation.

LEVEL

Algebra 2 or Math 3 as a review of tangent lines or in a unit on key features of function graphs

Geometry or Math 2 in a unit including secant and tangent lines

Algebra 1 or Math 1 as an extension of a unit on average rate of change

MODULE/CONNECTION TO AP*

Rate of Change: Average and Instantaneous

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MODALITY

NMSI emphasizes using multiple representations to connect various approaches to a situation in order to increase student understanding. The lesson provides multiple strategies and models for using those representations indicated by the darkened points of the star to introduce, explore, and reinforce mathematical concepts and to enhance conceptual understanding.

P – Physical
V – Verbal
A – Analytical
N – Numerical
G – Graphical

TEACHER PAGES

COMMON CORE STATE STANDARDS FOR MATHEMATICAL CONTENT

This lesson addresses the following Common Core State Standards for Mathematical Content. The lesson requires that students recall and apply each of these standards rather than providing the initial introduction to the specific skill. The star symbol (★) at the end of a specific standard indicates that the high school standard is connected to modeling.

Targeted Standards

F-IF.6: Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
See questions 1-4 , 6-8

Reinforced/Applied Standards

F-IF.4: For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
See questions 5, 8

COMMON CORE STATE STANDARDS FOR MATHEMATICAL PRACTICE

These standards describe a variety of instructional practices based on processes and proficiencies that are critical for mathematics instruction. NMSI incorporates these important processes and proficiencies to help students develop knowledge and understanding and to assist them in making important connections across grade levels. This lesson allows teachers to address the following Common Core State Standards for Mathematical Practice.

MP.2: Reason abstractly and quantitatively.
Students connect a verbal description, a graph, and a table of values for an object in motion and use the graph and table to interpret specific aspects of the object's motion.

MP.3: Construct viable arguments and critique the reasoning of others.
In question 5, students defend their conclusions about Susan's possible speeds.

FOUNDATIONAL SKILLS

The following skills lay the foundation for concepts included in this lesson:

- Calculate the slope between two points
- Interpret slope from a graph

ASSESSMENTS

The following types of formative assessments are embedded in this lesson:

- Students engage in independent practice.
- Students apply knowledge to a new situation.

The following assessments are located on our website:

- Rate of Change: Average and Instantaneous – Algebra 1 Free Response Questions
- Rate of Change: Average and Instantaneous – Algebra 1 Multiple Choice Questions
- Rate of Change: Average and Instantaneous – Geometry Free Response Questions
- Rate of Change: Average and Instantaneous – Geometry Multiple Choice Questions
- Rate of Change: Average and Instantaneous – Algebra 2 Free Response Questions
- Rate of Change: Average and Instantaneous – Algebra 2 Multiple Choice Questions

MATERIALS AND RESOURCES

- Student Activity pages
- Straightedges

TEACHING SUGGESTIONS

Begin this lesson by asking two students to simultaneously walk parallel paths across the room, with one student keeping a constant speed and the other student speeding up and slowing down, so that both students arrive at the "finish line" at the same time. Note that both students walked at the same average speed over the interval of the walk, but that their speeds within the walk's interval were not always the same. Students should apply these observations to questions 1 – 4.

Questions 5 – 7 introduce the concept that the slope of a non-linear function at a particular point is equal to the slope of a short line segment that is tangent to the function at that point. Students will probably remember that a line tangent to a circle touches the circle in only one point while a secant line intersects the circle at two points. The physical activity of drawing the secant lines and the tangent lines helps the visual and tactile learner understand the differences in the average rate of change and the instantaneous rate of change of non-linear functions.

Questions 5 and 6 foreshadow the Mean Value Theorem from calculus which refers to the mean (or average) rate of change of a function on an interval. This theorem states that, for a smooth continuous function, there must be a point within the interval where the instantaneous rate of change is equal to the average rate of change over that closed interval. So for Susan's graph, there must be at least one point during the time period where her instantaneous velocity is equal to her average velocity over the entire time period.

After students experience drawing tangent line segments to approximate the rate of change of a function, they apply this skill to analyze the distance-time graph of an object moving along a horizontal line in question 8. The average rate of change of a position function over a specified time interval indicates the average velocity over that interval. When calculating the slope over an interval, use the appropriate units to help students develop an understanding of this concept. To approximate the instantaneous rate of change at a point, students will draw a short tangent line segment and approximate its slope. By considering the sign and magnitude of the instantaneous rate of change in position with respect to time, students answer questions about the direction of the movement and the speed of the object. A positive velocity indicates that the motion is in a positive direction (away from the wall in this example). Negative velocity indicates motion in a negative direction (toward the wall in this example). In discussing speed, clarify for students that speed measures how fast the object is moving without regard to its direction and that speed is the absolute value of velocity.

You may wish to support this activity with TI-Nspire™ technology. See Working with Fractions and Decimals in the NMSI TI-Nspire Skill Builders.

Suggested modifications for additional scaffolding include the following:

1 Insert a fill-in-the-blank calculation with units provided for the dimensional analysis required to calculate the distance from home at the end of 20 minutes. (See the answer key for an example.) Edit the graph to indicate the starting point $\left(0, 2\frac{1}{3}\right)$.

2 Supply an example of a possible graph of Susan's drive before having the student draw a different possible graph.

5 Modify a sample graph from question 2 by drawing a secant line between two points to demonstrate that the slope is greater than 35 on that particular interval.

NMSI CONTENT PROGRESSION CHART

In the spirit of NMSI's goal to connect mathematics across grade levels, a Content Progression Chart for each module demonstrates how specific skills build and develop from sixth grade through pre-calculus in an accelerated program that enables students to take college-level courses in high school, using a faster pace to compress content. In this sequence, Grades 6, 7, 8, and Algebra 1 are compacted into three courses. Grade 6 includes all of the Grade 6 content and some of the content from Grade 7, Grade 7 contains the remainder of the Grade 7 content and some of the content from Grade 8, and Algebra 1 includes the remainder of the content from Grade 8 and all of the Algebra 1 content.

The complete Content Progression Chart for this module is provided on our website and at the beginning of the training manual. This portion of the chart illustrates how the skills included in this particular lesson develop as students advance through this accelerated course sequence.

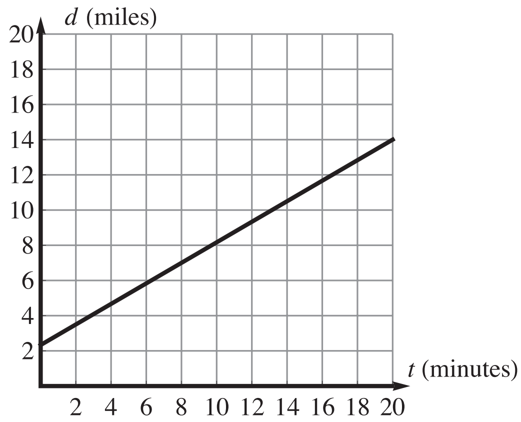
6th Grade Skills/Objectives	7th Grade Skills/Objectives	Algebra 1 Skills/Objectives	Geometry Skills/Objectives	Algebra 2 Skills/Objectives	Pre-Calculus Skills/Objectives
From graphical or tabular data or from a stated situation presented in paragraph form, calculate or compare the average rates of change and interpret the meaning.	From graphical or tabular data or from a stated situation presented in paragraph form, calculate or compare the average rates of change and interpret the meaning.	From graphical or tabular data or from a stated situation presented in paragraph form, calculate or compare the average rates of change and interpret the meaning.	From graphical or tabular data or from a stated situation presented in paragraph form, calculate or compare the average rates of change and interpret the meaning.	From graphical or tabular data or from a stated situation presented in paragraph form, calculate or compare the average rates of change and interpret the meaning.	From graphical or tabular data or from a stated situation presented in paragraph form, calculate or compare the average rates of change and interpret the meaning.
			Recognize intervals of functions with the same average rate of change.	Recognize intervals of functions with the same average rate of change.	Recognize intervals of functions with the same average rate of change.
			Compare average rates of change on different intervals in a table or graph.	Compare average rates of change on different intervals in a table or graph.	Compare average rates of change on different intervals in a table or graph.
			Estimate and/or compare instantaneous rates of change at a point based on the slopes of the tangent lines.	Estimate and/or compare instantaneous rates of change at a point based on the slopes of the tangent lines.	Estimate and/or compare instantaneous rates of change at a point based on the slopes of the tangent lines.

Average Rate of Change vs. Instantaneous Rate of Change

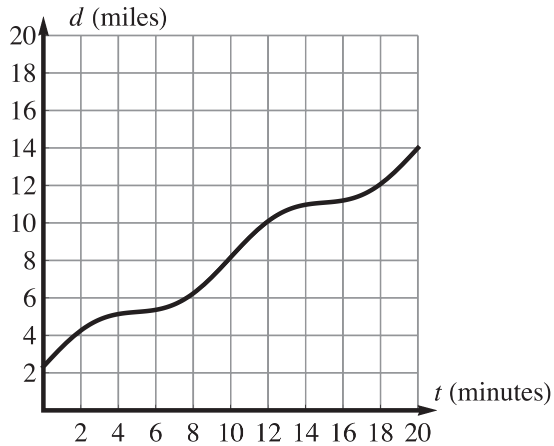
Answers

1. $d = 2\frac{1}{3}\text{ mi} + 35\frac{\text{mi}}{\text{hr}}\left(\frac{20\text{ min}}{1}\right)\left(\frac{1\text{ hr}}{60\text{ min}}\right) = 2\frac{1}{3}\text{mi} + 11\frac{2}{3}\text{mi}$

   $d = 14$ miles



2.



The graph is one of many possible examples. The only requirement is that the instantaneous rate of change is never less than or equal to 0 unless Susan stops or drives back toward home.

3. $\dfrac{14\text{ mi} - 2\frac{1}{3}\text{mi}}{\frac{1}{3}\text{ hr}} = 35\frac{\text{mi}}{\text{hr}}$; they traveled the same distance in the same amount of time.

4. $35\frac{\text{mi}}{\text{hr}}$; it is the same.

5. Yes; answers vary, but at least one secant line drawn between two relatively close points should have a slope greater than 35.

6. Answers vary for x and y, but slope should be $35\frac{\text{mi}}{\text{hr}}$ or $\frac{7}{12}\frac{\text{mi}}{\text{min}}$, so the short line segment drawn should be parallel to the original line.

7.

Rate of Change at Point	Letter
–2	B, C
0	A, D, E
3	P
6	F
15	G

8. a. At 0.13 seconds, the object is 27.5 inches from the wall. At 2 seconds, the object is 19 inches from the wall.

b. At points D and V, the instantaneous rate of change is negative. This negative velocity indicates that the object is moving toward the wall.

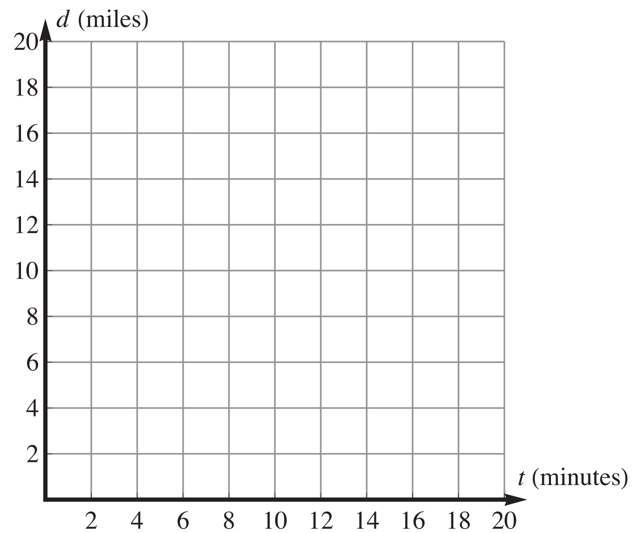
c. $(0.13, 0.87)$ and $(3.02, 4.1)$. A positive slope means the object is moving away from the wall. Even though this is a graph of a position function, when you determine the slope, you are finding how the position is changing with respect to time, which is the velocity.

d. The object is stopped at points B, K, and Z. The slopes of the tangent lines are zero at these points.

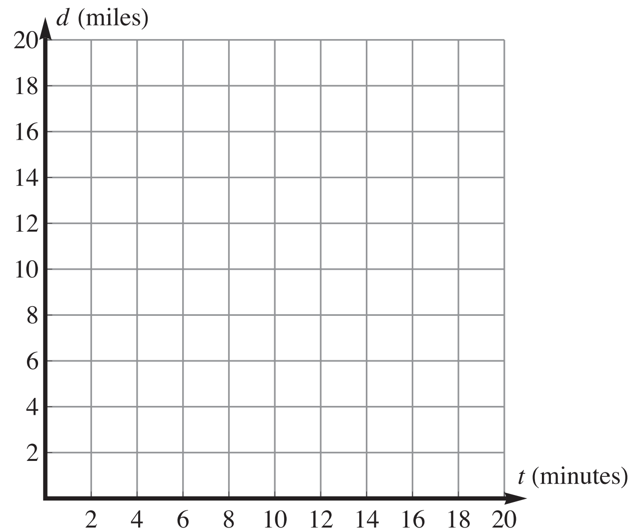
e. Z, G, D. The speed is determined by the magnitude (or absolute value) of the slope of the tangent line segment. The slope at Z is zero, making it the smallest. Without regard to the direction, the tangent line at G is less steep than the tangent line at D.

Average Rate of Change vs. Instantaneous Rate of Change

1. Beginning $2\frac{1}{3}$ miles from home, Jonathan drove away from home at a constant rate for 20 minutes. If his constant rate is 35 miles per hour, how far is he from home at the end of the 20 minutes? Draw a graph to model his distance from home during the 20 minute time period.



2. Susan, Jonathan's sister, also drove away from home beginning $2\frac{1}{3}$ miles from home and following the same path as Jonathan. Susan kept varying her velocity by frequently speeding up and slowing down. She arrived at the same location as Jonathan at the end of 20 minutes. To model Susan's distance from home during the 20 minute time period, draw a smooth curve without any sharp corners.



3. Calculate the average velocity for both drivers by calculating the change in position divided by the change in time. These two calculations have the same value; explain why this makes sense. Compare your answers to the rate given in question 1.

4. On the graph showing Susan's position, draw the line segment connecting the point at $t = 0$ and the point at $t = 20$. What is the slope of this line segment and what are the units for the slope? How does this slope compare to the slope of the line that modeled Jonathan's distance from home?

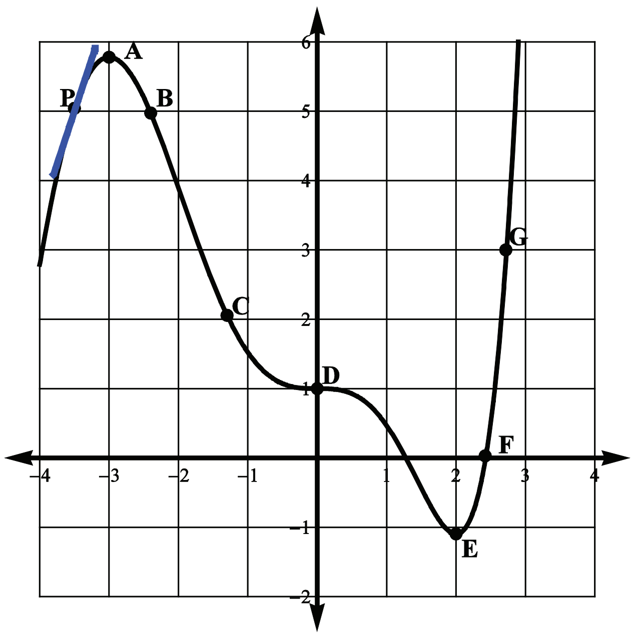
5. If the speed limit over the entire path is 35 miles per hour, did Susan ever drive over the speed limit? Explain your answer by referring to Susan's graph.

6. For non-linear position functions, the exact velocity at a particular time, called instantaneous rate of change or instantaneous velocity, cannot be calculated precisely without the tools of calculus. However, the velocity can be estimated by approximating the slope of a short line segment drawn tangent to the curve at the particular time. On the graph showing Susan's distance, locate at least one time when Susan's instantaneous velocity has the same value as the average velocity.

   (Position a straightedge on the graph so that it is parallel to the line segment drawn on the curve in question 4. Move the straightedge around on the graph keeping the slope of the straightedge fixed. When the straightedge appears to be tangent to the curve, mark the point(s) and sketch a short segment tangent to the curve. At these point(s), the instantaneous velocity is the same as the average velocity.)

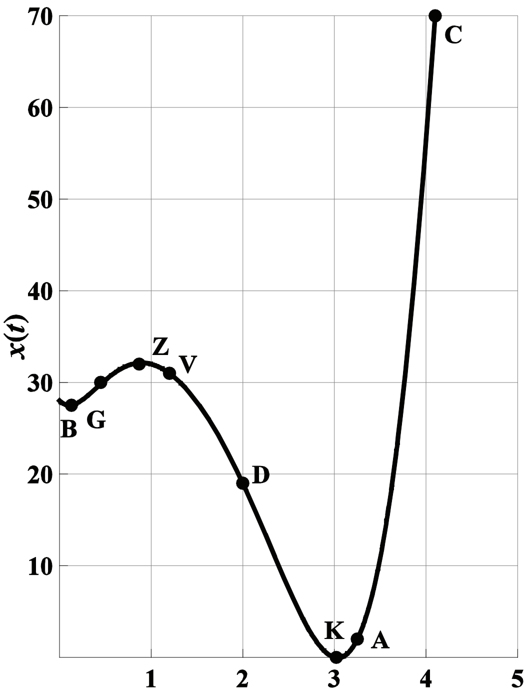
7. Using the function $g(x)$ shown in the graph, draw a small tangent line segment at each labeled point. A small tangent line is drawn at point P as a sample. There is not enough information to draw the segments perfectly, so sketches may vary slightly. Match the slope at each labeled point on the curve with an approximate rate of change value in the table. The slope at each point is called the "instantaneous rate of change" at a point because it is the rate of change at that one instant in time.

Hint: The slope may be the same at different places along the graph.



Rate of Change at Point	Letter
–2	
0	
3	p
6	
15	

8. The graph represents the position $x(t)$ in inches of an object that is moving along a line extending perpendicularly from a wall at a given time, t, measured in seconds. The distance between the object and the wall is indicated on the vertical axis, while time is measured on the horizontal axis.



Point	t	$x(t)$
B	0.13	27.5
G	0.45	30
Z	0.87	32
V	1.2	31
D	2.0	19
K	3.02	0
A	3.25	2
C	4.1	70

a. What do the coordinates of B (0.13, 27.5) and D (2, 19) represent in the context of this situation?

b. Mark small tangent line segments on each of the points that are named. Using these tangent segments, for which point(s) is the instantaneous rate of change negative? What do you know about the motion of the object if the instantaneous rate of change is negative?

c. Observing the tangent segments, over which time intervals is the object moving away from the wall? What do the slopes of these line segments mean in the context of the position function?

d. At which point(s) has the object stopped moving? Describe the slope of the tangent line(s) at the point(s).

e. Speed indicates how fast an object is moving without regard to direction. Order the speeds at the following points from least to greatest: D, G, Z. Explain your reasoning.