Area and Volume Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

Multiple Choice

1. C (1993 AB30)

   Each slice is a disk whose volume is given by $\pi r^2 \Delta x$ where $r = \sqrt{x}$.

   $$V = \pi\int_0^3 \left(\sqrt{x}\right)^2 \, dx = \pi\int_0^3 x \, dx = \frac{\pi}{2} x^2 \Big|_0^3 = \frac{9\pi}{2}$$

2. C (1973 AB15/BC15)

   $$\int_{0^{`}}^{2} e^{\frac{x}{2}} \, dx = 2\int_{0^{`}}^{2} \frac{1}{2} e^{\frac{x}{2}} \, dx = 2e^{\frac{x}{2}}\Big|_0^2 = 2(e-1)$$

3. C (1969 AB13)

   $$\int_{-\frac{\pi}{2}}^{k} \cos x \, dx = 3\int_{k}^{\frac{\pi}{2}} \cos x \, dx$$

   $$\sin k - \sin\left(-\frac{\pi}{2}\right) = 3(\sin\frac{\pi}{2} - \sin k)$$

   $$\sin k + 1 = 3 - 3\sin k$$

   $$4\sin k = 2$$

   $$\sin k = \frac{1}{2}$$

   $$k = \frac{\pi}{6}$$

4. B (2003 AB86)

   $$V_{slice} = s^2 \, dx$$

   $$V_{total} = \int_0^1 s^2 \, dx$$

   $$= \int_0^1 (3 - \arctan x)^2 \, dx$$

   $$= 6.612$$

5. B (2008 AB83/BC83)

Graph the curves to see that they intersect at $x = 1,\ 2,$ and 5.

Let $f(x) = x^3 - 8x^2 + 18x - 5$ and $g(x) = x + 5$.

Area = $\int_1^2 (f(x) - g(x))\, dx + \int_2^5 (g(x) - f(x))dx = 11.833$

Or Area = $\int_1^5 \left| f(x) - g(x) \right| dx = 11.833$

6. C (2003 AB77)

$\int_{-3}^{3} \left(f(x) + 1 \right) dx = \int_{-3}^{3} f(x)\, dx + \int_{-3}^{3} 1\, dx$

$= -2 + 2 - 2 + 6$

$= 4$

Free Response

7. (2010B AB1/BC1)

	1: Correct limits in an integral in (a), (b), or (c)
(a) $\int_0^2 \left(6 - 4\ln\left(3 - x\right)\right) dx = 6.816$ or 6.817	$2 \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(b) $\pi \int_0^2 \left(\left(8 - 4\ln\left(3 - x\right)\right)^2 - \left(8 - 6\right)^2 \right) dx$ $= 168.179$ or 168.180	$3 \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$
(c) $\int_0^2 \left(6 - 4\ln\left(3 - x\right)\right)^2 dx = 26.266$ or 26.267	$3 \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

8. (2011B AB3)

(a) $\text{Area} = \int_0^4 \sqrt{x}\,dx + \frac{1}{2}\cdot 2\cdot 2 = \frac{2}{3}x^{3/2}\Big\vert_{x=0}^{x=4} + 2 = \frac{22}{3}$	$3\begin{cases} 1: \text{ integral} \\ 1: \text{ antiderivative} \\ 1: \text{ answer} \end{cases}$
(b) $y = \sqrt{x} \Rightarrow x = y^2$ $y = 6 - x \Rightarrow x = 6 - y$ $\text{Width} = (6 - y) - y^2$ $\text{Volume} = \int_0^2 2y\left(6 - y - y^2\right)dy$	$3\begin{cases} 2: \text{ integrand} \\ 1: \text{ answer} \end{cases}$
(c) $g'(x) = -1$ Thus a line perpendicular to the graph of g has slope of 1. $f'(x) = \frac{1}{2\sqrt{x}}$ $\frac{1}{2\sqrt{x}} = 1 \Rightarrow x = \frac{1}{4}$ The point P has coordinates $\left(\frac{1}{4}, \frac{1}{2}\right)$.	$3\begin{cases} 1: \ f'(x) \\ 1: \text{ equation} \\ 1: \text{ answer} \end{cases}$

9. (2007 AB1/BC1)

$\frac{20}{1+x^2} =$ when $x = \pm 3$	1: correct limits in an integral in (a), (b), or (c)
(a) Area $= \int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right) dx =$ 37.961 or 37.962	$2 \begin{cases} 1: & \text{integrand} \\ 1: & \text{answer} \end{cases}$
(b) Volume $= \pi \int_{-3}^{3} \left(\left(\frac{20}{1+x^2} \right)^2 - 2^2 \right) dx =$ 1871.190	$3 \begin{cases} 2: & \text{integrand} \\ 1: & \text{answer} \end{cases}$
(c) Volume $= \frac{\pi}{2} \int_{-3}^{3} \left(\frac{1}{2} \left(\frac{20}{1+x^2} - 2 \right) \right)^2 dx$ $= \frac{\pi}{8} \int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right)^2 dx = 174.268$	$3 \begin{cases} 2: & \text{integrand} \\ 1: & \text{answer} \end{cases}$

10. (2004 Form B AB6/BC6)

(a) $\int_0^1 x^n\,dx = \left.\dfrac{x^{n+1}}{n+1}\right\|_0^1 = \dfrac{1}{n+1}$	$2\begin{cases}1: \text{ antiderivative of } x^n \\ 1: \text{ answer}\end{cases}$
(b) Let b be the length of the base of triangle T. $\dfrac{1}{b}$ is the slope of line ℓ, which is n $\text{Area}(T) = \dfrac{1}{2}b(1) = \dfrac{1}{2n}$	$3\begin{cases}1: \text{ slope of line } \ell \text{ is } n \\ 1: \text{ base of } T \text{ is } \dfrac{1}{n} \\ 1: \text{ shows area is } \dfrac{1}{2n}\end{cases}$
(c) $\text{Area}(S) = \int_0^1 x^n\,dx - \text{Area}(T)$ $= \dfrac{1}{n+1} - \dfrac{1}{2n}$ $\dfrac{d}{dn}\text{Area}(S) = -\dfrac{1}{(n+1)^2} + \dfrac{1}{2n^2} = 0$ $2n^2 = (n+1)^2$ $\sqrt{2}n = (n+1)$ $n = \dfrac{1}{\sqrt{2}-1} = 1+\sqrt{2}$	$4\begin{cases}1: \text{ area of } S \text{ in terms of } n \\ 1: \text{ derivative} \\ 1: \text{ sets derivative equal to 0} \\ 1: \text{ solves for } n\end{cases}$

11. (2008 AB1/BC1)

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	$3\begin{cases} 1: \text{ limits} \\ 1: \text{ integrand} \\ 1: \text{ answer} \end{cases}$
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	$2\begin{cases} 1: \text{ limits} \\ 1: \text{ integrand} \end{cases}$
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	$2\begin{cases} 1: \text{ integrand} \\ 1: \text{ limits} \end{cases}$
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	$2\begin{cases} 1: \text{ integrand} \\ 1: \text{ limits} \end{cases}$