2.4 Rates of Change and Tangent Lines

What you will learn about...

- Average Rates of Change
- Tangent to a Curve
- Slope of a Curve
- Normal to a Curve
- Speed Revisited

and why...

The tangent line determines the direction of a body's motion at every point along its path.

Average Rates of Change

We encounter average rates of change in such forms as average speed (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

SOLUTION

Since $f(1) = 0$ and $f(3) = 24$, the average rate of change over the interval $[1, 3]$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.$$

Now Try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 2.27 shows how the number of fruit flies *(Drosophila)* grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

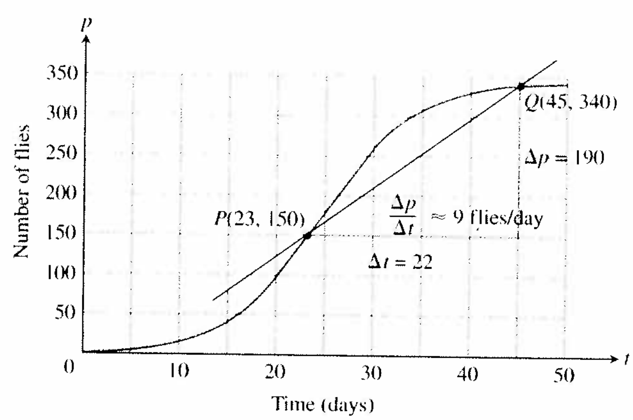


Figure 2.27 Growth of a fruit fly population in a controlled experiment.
Source: Elements of Mathematical Biology. (Example 2)

Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

Marjorie Lee Browne (1914–1979)

When Marjorie Browne graduated from the University of Michigan in 1949, she was one of the first two African American women to be awarded a Ph.D. in Mathematics. Browne went on to become chairperson of the mathematics department at North Carolina Central University, and succeeded in obtaining grants for retraining high school mathematics teachers.

EXAMPLE 2 Growing *Drosophila* in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 2.27 to compute the average rate of change and the slope of the secant line PQ.

SOLUTION

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of $340 - 150 = 190$ flies in $45 - 23 = 22$ days.

The average rate of change in the population p from day 23 to day 45 was

$$\textit{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day},$$

or about 9 flies per day.

continued

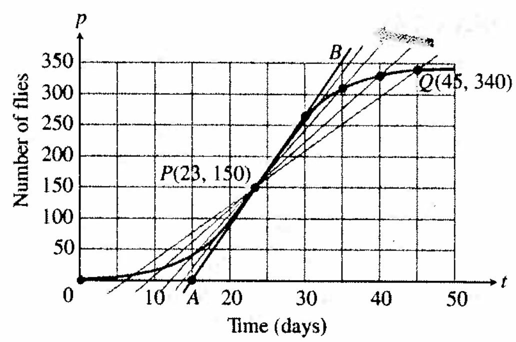
This average rate of change is also the slope of the secant line through the two points P and Q on the population curve. We can calculate the slope of the secant PQ from the coordinates of P and Q.

$$\textit{Secant slope:}\ \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}$$

Now Try Exercise 7.

As suggested by Example 2, *we can always think of an average rate of change as the slope of a secant line.*

In addition to knowing the average rate at which the population grew from day 23 to day 45, we may also want to know how fast the population was growing on day 23 itself. To find out, we can watch the slope of the secant PQ change as we back Q along the curve toward P. The results for four positions of Q are shown in Figure 2.28.



(a)

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	(340 − 150)/(45 − 23) ≈ 8.6
(40, 330)	(330 − 150)/(40 − 23) ≈ 10.6
(35, 310)	(310 − 150)/(35 − 23) ≈ 13.3
(30, 265)	(265 − 150)/(30 − 23) ≈ 16.4

(b)

Figure 2.28 (a) Four secants to the fruit fly graph of Figure 2.27, through the point $P(23, 150)$. (b) The slopes of the four secants.

In terms of geometry, what we see as Q approaches P along the curve is this: The secant PQ approaches the tangent line AB that we drew by eye at P. This means that within the limitations of our drawing, the slopes of the secants approach the slope of the tangent, which we calculate from the coordinates of A and B to be

$$\frac{350 - 0}{35 - 15} = 17.5 \text{ flies/day.}$$

In terms of population, what we see as Q approaches P is this: The average growth rates for increasingly smaller time intervals approach the slope of the tangent to the curve at P (17.5 flies per day). The slope of the tangent line is therefore the number we take as the rate at which the fly population was growing on day $t = 23$.

Tangent to a Curve

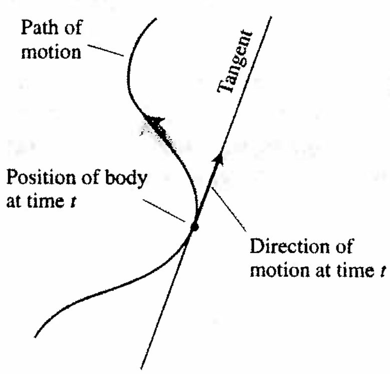
The moral of the fruit fly story would seem to be that we should define the rate at which the value of the function $y = f(x)$ is changing with respect to x at any particular value $x = a$ to be the slope of the tangent to the curve $y = f(x)$ at $x = a$. But how are we to define the tangent line at an arbitrary point P on the curve and find its slope from the formula $y = f(x)$? The problem here is that we know only one point. Our usual definition of slope requires two points.

The solution that mathematician Pierre Fermat found in 1629 proved to be one of that century's major contributions to calculus. We still use his method of defining tangents to produce formulas for slopes of curves and rates of change:

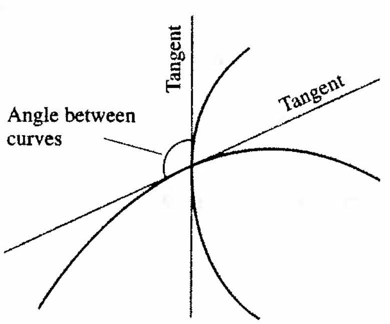
1. We start with what we can calculate, namely, the slope of a secant through P and a point Q nearby on the curve.

Why Find Tangents to Curves?

In mechanics, the tangent determines the direction of a body's motion at every point along its path.



In geometry, the tangents to two curves at a point of intersection determine the angle at which the curves intersect.



In optics, the tangent determines the angle at which a ray of light enters a curved lens (more about this in Section 4.2). The problem of how to find a tangent to a curve became the dominant mathematical problem of the early 17th century, and it is hard to overestimate how badly the scientists of the day wanted to know the answer. Descartes went so far as to say that the problem was the most useful and most general problem not only that he knew but that he had any desire to know.

2. We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.

3. We define the *slope of the curve* at P to be this number and define the *tangent to the curve* at P to be the line through P with this slope.

EXAMPLE 3 Finding Slope and Tangent Line

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

SOLUTION

We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$ on the curve (Figure 2.29).

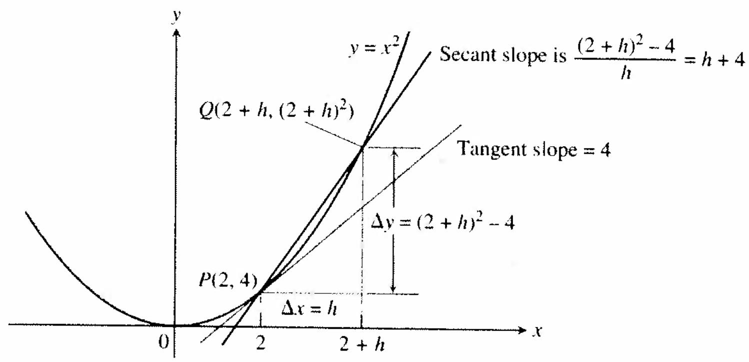


Figure 2.29 The slope of the tangent to the parabola $y = x^2$ at $P(2, 4)$ is 4.

We then write an expression for the slope of the secant line and find the limiting value of this slope as Q approaches P along the curve.

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 4}{h}$$
$$= \frac{h^2 + 4h + 4 - 4}{h}$$
$$= \frac{h^2 + 4h}{h} = h + 4$$

The limit of the secant slope as Q approaches P along the curve is

$$\lim_{Q \to P} (\text{secant slope}) = \lim_{h \to 0} (h + 4) = 4.$$

Thus, the slope of the parabola at P is 4.

The tangent to the parabola at P is the line through $P(2, 4)$ with slope $m = 4$.

$$y - 4 = 4(x - 2)$$
$$y = 4x - 8 + 4$$
$$y = 4x - 4$$

Now Try Exercise 11 (a, b).

Pierre de Fermat (1601–1665)

The dynamic approach to tangency, invented by Fermat in 1629, proved to be one of the 17th century's major contributions to calculus. Fermat, a skilled linguist and one of his century's greatest mathematicians, tended to confine his writing to professional correspondence and to papers written for personal friends. He rarely wrote completed descriptions of his work, even for his personal use. His name slipped into relative obscurity until the late 1800s, and it was only from a four-volume edition of his works published at the beginning of this century that the true importance of his many achievements became clear.

Slope of a Curve

To find the tangent to a curve $y = f(x)$ at a point $P(a, f(a))$ we use the same dynamic procedure. We calculate the slope of the secant line through P and a point $Q(a + h, f(a + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 2.30). If the limit exists, it is the slope of the curve at P and we define the tangent at P to be the line through P having this slope.

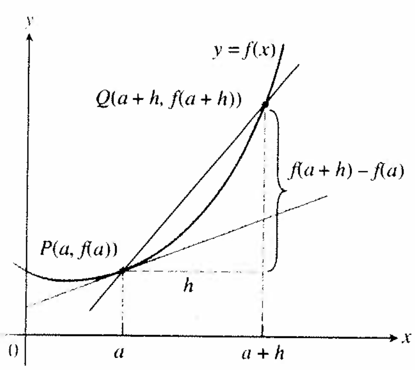


Figure 2.30 The tangent slope is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

DEFINITION Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

The **tangent line to the curve** at P is the line through P with this slope.

EXAMPLE 4 Exploring Slope and Tangent

Let $f(x) = 1/x$.

(a) Find the slope of the curve at $x = a$.

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ for different values of a?

SOLUTION

(a) The slope at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \cdot \frac{-h}{ha(a+h)}$$

$$= \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

(b) The slope will be $-1/4$ if

$$-\frac{1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4 \qquad \text{Multiply by } -4a^2.$$

$$a = \pm 2.$$

The curve has the slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 2.31).

(c) The slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep. We see this again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent becomes increasingly horizontal.

Now Try Exercise 19.

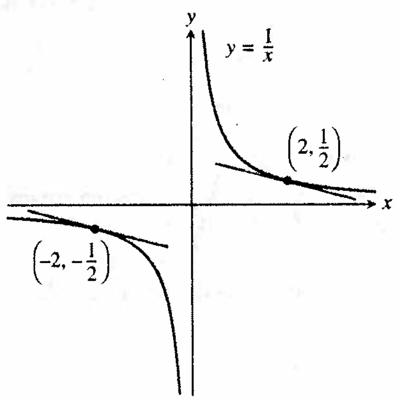


Figure 2.31 The two tangent lines to $y = 1/x$ having slope $-1/4$. (Example 4)

All of These Are the Same:

1. the slope of $y = f(x)$ at $x = a$
2. the slope of the tangent to $y = f(x)$ at $x = a$
3. the (instantaneous) rate of change of $f(x)$ with respect to x at $x = a$
4. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

The expression

$$\frac{f(a+h) - f(a)}{h}$$

is the **difference quotient of f at a.** Suppose the difference quotient has a limit as h approaches zero. If we interpret the difference quotient as a secant slope, the limit is the slope of both the curve and the tangent to the curve at the point $x = a$. If we interpret the difference quotient as an average rate of change, the limit is the function's rate of change with respect to x at the point $x = a$. This limit is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 3.

An Alternate Form

In Chapter 3, we will introduce the expression

$$\frac{f(x) - f(a)}{x - a}$$

as an important and useful alternate form of the **difference quotient of f at a.** (See Exercise 55.)

About the Word *Normal*

When analytic geometry was developed in the 17th century, European scientists still wrote about their work and ideas in Latin, the one language that all educated Europeans could read and understand. The Latin word *normalis,* which scholars used for *perpendicular,* became *normal* when they discussed geometry in English.

Normal to a Curve

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

EXAMPLE 5 Finding a Normal Line

Write an equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.

SOLUTION

The slope of the tangent to the curve at $x = 1$ is

$$\begin{aligned}\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{4-(1+h)^2-3}{h}\\ &= \lim_{h\to 0}\frac{4-1-2h-h^2-3}{h}\\ &= \lim_{h\to 0}\frac{-h(2+h)}{h} = -2.\end{aligned}$$

Thus, the slope of the normal is $1/2$, the negative reciprocal of -2. The normal to the curve at $(1, f(1)) = (1, 3)$ is the line through $(1, 3)$ with slope $m = 1/2$.

$$\begin{aligned}y - 3 &= \frac{1}{2}(x-1)\\ y &= \frac{1}{2}x - \frac{1}{2} + 3\\ y &= \frac{1}{2}x + \frac{5}{2}\end{aligned}$$

You can support this result by drawing the graphs in a square viewing window.

Now Try Exercise 11 (c, d).

Speed Revisited

The function $y = 16t^2$ that gave the distance fallen by the rock in Example 1, Section 2.1, was the rock's *position function.* A body's average speed along a coordinate axis (here, the y-axis) for a given period of time is the average rate of change of its *position* $y = f(t)$. Its *instantaneous speed* at any time t is the **instantaneous rate of change** of position with respect to time at time t, or

$$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}.$$

We saw in Example 1, Section 2.1, that the rock's instantaneous speed at $t = 2$ sec was 64 ft/sec.

Particle Motion

We only have considered objects moving in one direction in this chapter. In Chapter 3, we will deal with more complicated motion.

EXAMPLE 6 Finding Instantaneous Rate of Change

Find

$$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

for the function $f(t) = 2t^2 - 1$ at $t = 2$. Interpret the answer if $f(t)$ represents a position function in feet of an object at time t seconds.

continued

SOLUTION

$$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{2(2+h)^2-1-(2\cdot 2^2-1)}{h}$$
$$= \lim_{h\to 0}\frac{8+8h+2h^2-1-7}{h}$$
$$= \lim_{h\to 0}\frac{8h+2h^2}{h}$$
$$= \lim_{h\to 0}(8+2h) = 8$$

The instantaneous rate of change of the object is 8 ft/sec.

Now Try Exercise 23.

EXAMPLE 7 Investigating Free Fall

Find the speed of the falling rock in Example 1, Section 2.1, at $t = 1$ sec.

SOLUTION

The position function of the rock is $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ sec was

$$\frac{f(1+h)-f(1)}{h} = \frac{16(1+h)^2-16(1)^2}{h} = \frac{16(h^2+2h)}{h} = 16(h+2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h\to 0} 16(h+2) = 32 \text{ ft/sec.}$$

Now Try Exercise 31.

Quick Review 2.4 *(For help, go to Section 1.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B.

1. $A(-5, 2)$, $B(3, 5)$
2. $A(1, 3)$, $B(a, b)$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3)$, $(5, -1)$
4. $(-3, -1)$, $(3, 3)$

In Exercises 5–9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= 3/2$
6. through $(1, 6)$ and $(4, -1)$
7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$
8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$
9. through $(-1, 3)$ and parallel to $2x + 3y = 5$
10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $5/3$?

Section 2.4 Exercises

In Exercises 1–6, find the average rate of change of the function over each interval.

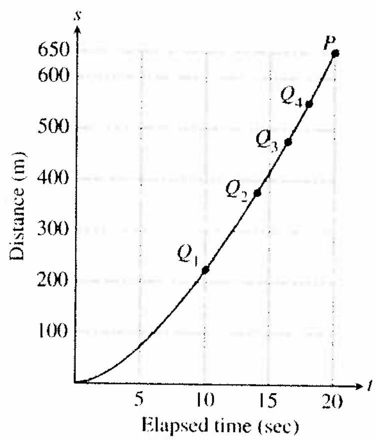
1. $f(x) = x^3 + 1$
   (a) $[2, 3]$ **(b)** $[-1, 1]$
2. $f(x) = \sqrt{4x + 1}$
   (a) $[0, 2]$ **(b)** $[10, 12]$
3. $f(x) = e^x$
   (a) $[-2, 0]$ **(b)** $[1, 3]$
4. $f(x) = \ln x$
   (a) $[1, 4]$ **(b)** $[100, 103]$
5. $f(x) = \cot x$
   (a) $[\pi/4, 3\pi/4]$ **(b)** $[\pi/6, \pi/2]$
6. $f(x) = 2 + \cos x$
   (a) $[0, \pi]$ **(b)** $[-\pi, \pi]$

In Exercises 7 and 8, a distance-time graph is shown.

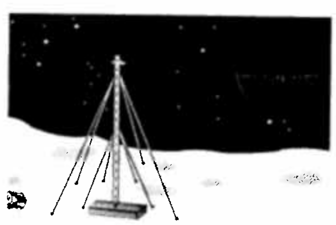
(a) Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table. What is the appropriate unit for these slopes?

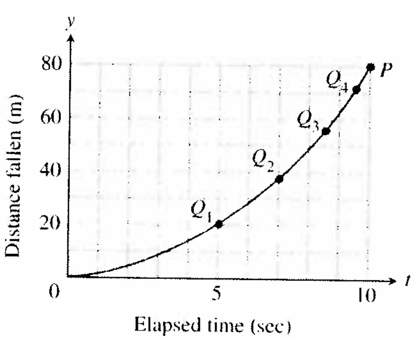
(b) Estimate the speed at point P.

7. ***Accelerating from a Standstill*** The figure shows the distance-time graph for a 1994 Ford® Mustang Cobra™ accelerating from a standstill.



8. ***Lunar Data*** The accompanying figure shows a distance-time graph for a wrench that fell from the top platform of a communication mast on the moon to the station roof 80 m below.





In Exercises 9–12, at the indicated point find

(a) the slope of the curve,

(b) an equation of the tangent, and

(c) an equation of the normal.

(d) Then draw a graph of the curve, tangent line, and normal line in the same square viewing window.

9. $y = x^2$ at $x = -2$

10. $y = x^2 - 4x$ at $x = 1$

11. $y = \dfrac{1}{x-1}$ at $x = 2$

12. $y = x^2 - 3x - 1$ at $x = 0$

In Exercises 13 and 14, find the slope of the curve at the indicated point.

13. $f(x) = |x|$ at **(a)** $x = 2$ **(b)** $x = -3$

14. $f(x) = |x - 2|$ at $x = 1$

In Exercises 15–18, determine whether the curve has a tangent at the indicated point. If it does, give its slope. If not, explain why not.

15. $f(x) = \begin{cases} 2 - 2x - x^2, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$ at $x = 0$

16. $f(x) = \begin{cases} -x, & x < 0 \\ x^2 - x, & x \geq 0 \end{cases}$ at $x = 0$

17. $f(x) = \begin{cases} 1/x, & x \leq 2 \\ \dfrac{4-x}{4}, & x > 2 \end{cases}$ at $x = 2$

18. $f(x) = \begin{cases} \sin x, & 0 \leq x < 3\pi/4 \\ \cos x, & 3\pi/4 \leq x \leq 2\pi \end{cases}$ at $x = 3\pi/4$

In Exercises 19–22, **(a)** find the slope of the curve at $x = a$.
(b) Writing to Learn Describe what happens to the tangent at $x = a$ as a changes.

19. $y = x^2 + 2$

20. $y = 2/x$

21. $y = \dfrac{1}{x-1}$

22. $y = 9 - x^2$

Find the instantaneous rate of change of the position function $y = f(t)$ in feet at the given time t in seconds.

23. $f(t) = 3t - 7, \quad t = 1$

24. $f(t) = 3t^2 + 2t, \quad t = 3$

25. $f(t) = \dfrac{t+1}{t}, \quad t = 2$

26. $f(t) = t^3 - 1, \quad t = 2$

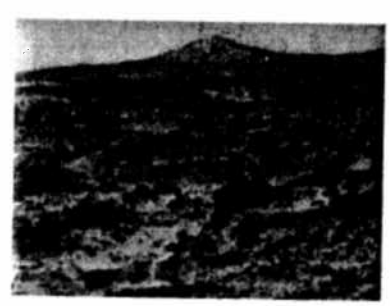
27. ***Free Fall*** An object is dropped from the top of a 100-m tower. Its height above ground after t sec is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

28. ***Rocket Launch*** At t sec after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec?

29. ***Area of Circle*** What is the rate of change of the area of a circle with respect to the radius when the radius is $r = 3$ in.?

30. ***Volume of Sphere*** What is the rate of change of the volume of a sphere with respect to the radius when the radius is $r = 2$ in.?

31. ***Free Fall on Mars*** The equation for free fall at the surface of Mars is $s = 1.86t^2$ m with t in seconds. Assume a rock is dropped from the top of a 200-m cliff. Find the speed of the rock at $t = 1$ sec.



32. ***Free Fall on Jupiter*** The equation for free fall at the surface of Jupiter is $s = 11.44t^2$ m with t in seconds. Assume a rock is dropped from the top of a 500-m cliff. Find the speed of the rock at $t = 2$ sec.

33. ***Horizontal Tangent*** At what point is the tangent to $f(x) = x^2 + 4x - 1$ horizontal?

34. ***Horizontal Tangent*** At what point is the tangent to $f(x) = 3 - 4x - x^2$ horizontal?

35. ***Finding Tangents and Normals***

(a) Find an equation for each tangent to the curve $y = 1/(x - 1)$ that has slope -1. (See Exercise 21.)

(b) Find an equation for each normal to the curve $y = 1/(x - 1)$ that has slope 1.

36. ***Finding Tangents*** Find the equations of all lines tangent to $y = 9 - x^2$ that pass through the point $(1, 12)$.

37. Table 2.2 gives the total amount of U.S. exported wheat products in metric tons for several years.

TABLE 2.2 U.S. Exported Wheat Products

Year	Exported Wheat Products (metric tons)
2000	844
2004	381
2005	313
2006	281
2007	448
2008	389

Source: U.S. Department of Agriculture, Economic Research Service, *Foreign Agricultural Trade of the United States, (FATUS)*, Table 820.

(a) Let $x = 0$ represent 2000, $x = 1$ represent 2001, and so forth. Make a scatter plot of the data.

(b) Let P represent the point corresponding to 2008, Q_1 the point corresponding to 2004, Q_2 the point corresponding to 2005, and Q_3 the point corresponding to 2007. Find the slope of the secant line PQ_i for $i = 1, 2, 3$.

38. Table 2.3 gives the amount of federal spending in billions of dollars for national defense for several years.

TABLE 2.3 National Defense Spending

Year	National Defense Spending ($ billions)
2003	404.8
2004	455.8
2005	495.3
2006	521.8
2007	551.3
2008	616.1
2009	690.3

Source: U.S. Office of Management and Budget, *Budget Authority by Function and Subfunction, Outlay by Function and Subfunction,* Table 492.

(a) Find the average rate of change in spending from 2003 to 2009.

(b) Find the average rate of change in spending from 2005 to 2008.

(c) Find the average rate of change in spending from 2008 to 2009.

(d) **Writing to Learn** Explain why someone might be hesitant to make predictions about the rate of change of national defense spending based on the data given in Table 2.3.

Standardized Test Questions

39. **True or False** If the graph of a function has a tangent line at $x = a$, then the graph also has a normal line at $x = a$. Justify your answer.

40. **True or False** The graph of $f(x) = |x|$ has a tangent line at $x = 0$. Justify your answer.

41. **Multiple Choice** If the line L tangent to the graph of a function f at the point $(2, 5)$ passes through the point $(-1, -3)$, what is the slope of L?

(A) $-3/8$ (B) $3/8$ (C) $-8/3$ (D) $8/3$ (E) undefined

42. **Multiple Choice** Find the average rate of change of $f(x) = x^2 + x$ over the interval $[1, 3]$.

(A) -5 (B) $1/5$ (C) $1/4$ (D) 4 (E) 5

43. **Multiple Choice** Which of the following is an equation of the tangent to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = -2x$ (B) $y = 2x$ (C) $y = -2x + 4$

(D) $y = -x + 3$ (E) $y = x + 3$

44. **Multiple Choice** Which of the following is an equation of the normal to the graph of $f(x) = 2/x$ at $x = 1$?

(A) $y = \frac{1}{2}x + \frac{3}{2}$ (B) $y = -\frac{1}{2}x$ (C) $y = \frac{1}{2}x + 2$

(D) $y = -\frac{1}{2}x + 2$ (E) $y = 2x + 5$

Explorations

In Exercises 45 and 46, complete the following for the function.

(a) Compute the difference quotient

$$\frac{f(1 + h) - f(1)}{h}.$$

(b) Use graphs and tables to estimate the limit of the difference quotient in part (a) as $h \to 0$.

(c) Compare your estimate in part (b) with the given number.

(d) **Writing to Learn** Based on your computations, do you think the graph of f has a tangent at $x = 1$? If so, estimate its slope. If not, explain why not.

45. $f(x) = e^x$, e

46. $f(x) = 2^x$, ln 4

Group Activity In Exercises 47–50, the curve $y = f(x)$ has a **vertical tangent** at $x = a$ if

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \infty$$

or if

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = -\infty.$$

In each case, the right- and left-hand limits are required to be the same: both $+\infty$ or both $-\infty$.

Use graphs to investigate whether the curve has a vertical tangent at $x = 0$.

47. $y = x^{2/5}$

48. $y = x^{3/5}$

49. $y = x^{1/3}$

50. $y = x^{2/3}$

Extending the Ideas

In Exercises 51 and 52, determine whether the graph of the function has a tangent at the origin. Explain your answer.

51. $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

52. $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

53. *Sine Function* Estimate the slope of the curve $y = \sin x$ at $x = 1$. (*Hint:* See Exercises 45 and 46.)

54. Consider the function f given in Example 1. Explain how the average rate of change of f over the interval $[3, 3 + h]$ is the same as the difference quotient of f at $a = 3$.

55. **(a)** Let $x = a + h$. Show algebraically how the difference quotient of f at a,

$$\frac{f(a + h) - f(a)}{h},$$

is equivalent to an alternate form given by

$$\frac{f(x) - f(a)}{x - a}.$$

(b) Writing to Learn Why do you think we discuss two forms of the difference quotient of f at a?

Quick Quiz for AP* Preparation: Sections 2.3 and 2.4

You may use a calculator with these problems.

1. Multiple Choice Which of the following values is the average rate of $f(x) = \sqrt{x + 1}$ over the interval $(0, 3)$?

(A) -3 **(B)** -1 **(C)** $-1/3$ **(D)** $1/3$ **(E)** 3

2. Multiple Choice Which of the following statements is false for the function

$$f(x) = \begin{cases} \frac{3}{4}x, & 0 \le x < 4 \\ 2, & x = 4 \\ -x + 7, & 4 < x \le 6 \\ 1, & 6 < x < 8? \end{cases}$$

(A) $\lim_{x \to 4} f(x)$ exists **(B)** $f(4)$ exists

(C) $\lim_{x \to 6} f(x)$ exists **(D)** $\lim_{x \to 8^-} f(x)$ exists

(E) f is continuous at $x = 4$

3. Multiple Choice Which of the following is an equation for the tangent line to $f(x) = 9 - x^2$ at $x = 2$?

(A) $y = \frac{1}{4}x + \frac{9}{2}$ **(B)** $y = -4x + 13$

(C) $y = -4x - 3$ **(D)** $y = 4x - 3$

(E) $y = 4x + 13$

4. Free Response Let $f(x) = 2x - x^2$.

(a) Find $f(3)$. **(b)** Find $f(3 + h)$.

(c) Find $\frac{f(3 + h) - f(3)}{h}$.

(d) Find the instantaneous rate of change of f at $x = 3$.

Chapter 2 Key Terms

average rate of change (p. 87)
average speed (p. 59)
composite of continuous functions (p. 82)
connected graph (p. 83)
Constant Multiple Rule for Limits (p. 61)
continuity at a point (p. 78)
continuous at an endpoint (p. 79)
continuous at an interior point (p. 79)
continuous extension (p. 81)
continuous function (p. 81)
continuous on an interval (p. 81)
difference quotient (p. 90)
Difference Rule for Limits (p. 61)
discontinuous (p. 79)
end behavior model (p. 74)
free fall (p. 91)
horizontal asymptote (p. 70)
infinite discontinuity (p. 80)
instantaneous rate of change (p. 91)
instantaneous speed (p. 91)
intermediate value property (p. 83)
Intermediate Value Theorem for Continuous Functions (p. 83)
jump discontinuity (p. 80)
left end behavior model (p. 74)
left-hand limit (p. 64)
limit of a function (p. 60)
normal to a curve (p. 91)
oscillating discontinuity (p. 80)
point of discontinuity (p. 79)
Power Rule for Limits (p. 71)
Product Rule for Limits (p. 61)
Properties of Continuous Functions (p. 82)
Quotient Rule for Limits (p. 61)
removable discontinuity (p. 80)
right end behavior model (p. 74)
right-hand limit (p. 64)
Sandwich Theorem (p. 65)
secant to a curve (p. 87)
slope of a curve (p. 89)
Sum Rule for Limits (p. 61)
tangent line to a curve (p. 88)
two-sided limit (p. 64)
vertical asymptote (p. 72)
vertical tangent (p. 94)